ENGINEERING ECONOMICS

FIRST PRINCIPLES

BY

JOHN CHARLES LOUNSBURY FISH

MEMBER AMERICAN SOCIETY OF CIVIL ENGINEERS; MEMBER AMERICAN RAILWAY ENGINEERING ASSOCIATION; SOMETIME DIVISION ENGINEER LAKE SHORE AND MICHIGAN SOUTHERN RAILWAY; PROFESSOR OF RAILROAD ENGINEERING LELAND STANFORD JUNIOR UNIVERSITY; AUTHOR OF "EARTHWORK HAUL AND OVERHAUL," ETC

FIRST EDITION

McGRAW-HILL BOOK COMPANY, INC.
239 WEST 39TH STREET, NEW YORK
6 BOUVERIE STREET, LONDON, E. C.
1915

THE MAPLE PRESS YORK PA

PREFACE

Every engineering structure, with few exceptions, is first suggested by economic requirements; and the design of every part, excepting few, and of the whole is finally judged from the economic standpoint.

It is therefore apparent that the so-called principles of design are subordinate to the principles which underlie economic judgment.

This important fact usually escapes the student of engineering because, while he may have seen hundreds of books on the principles of design and his time is largely employed in studying these principles and their application, he has seen not one book devoted to the principles which underlie economic judgment, and his books and his instructors merely mention these in passing.

It is true that during the past dozen years, chiefly as a result of the writings of such men as H. P. Gillette and F. W. Taylor, interest in the economic side of engineering has greatly widened; so that we now see the word "economics" in titles of technical articles, chapters, and even books.

Most of these works, however, assume the reader to be familiar with fundamental economic principles.

The present work was undertaken with the belief that to the engineer a working knowledge of first principles is as essential in the economics as in the mechanics of structures; and that special study and drill in the application of principles is as advantageous in the one case as in the other.

The book is intended to meet the first needs of the student, and to render effective service in the office. It is hoped that it will facilitate the introduction of formal instruction in engineering economics in the engineering schools, and assist sound engineering practice.

For the matter and arrangement of the book, the reader is referred to the table of contents, to the index, and to the notes to be found immediately below chapter heads.

Three years' use of the manuscript in the classroom seems to indicate that the student should take up the sections of the book

in the following order: §§ 1–28, 34–37, 50, 71–74, 76–78, 81, 83, 106–134, 38–70, 75, 79, 80, 82, 84–105. These sections can be covered in approximately thirty lessons. In connection with the lessons the student should be given many problems to solve, in order that he may become self-reliant in the application of principles.

Quotations from Waddell's "De Pontibus," Foster's "Engineering Valuation of Public Utilities and Factories," Gillette's "Cost Data," and Floy's "Valuation of Public Utility Properties" have been inserted with the kind permission of Dr. J. A. L. Waddell, D. Van Nostrand Company, Myron C. Clark Publishing Company, and McGraw-Hill Book Company, Inc., respectively.

For illustrative examples in Chapter X, I am under obligations to Messrs. J. F. Byxbee, J. B. Cox, H. H. Hall, G. J. Hoffman, and Charles Moser, and to Professors W. F. Durand, D. M. Folsom, J. H. Foss, and G. H. Marx.

I am indebted to Messrs. F. G. Athearn, A. C. Sandstrom, R. L. Vaughn, and Arthur Taylor for valued suggestions; to Mr. J. B. Cox for major and minor improvements which he effected in the course of critically examining the text, checking the formulas and examples, and proof-reading; and to Mr. H. H. Hall, whose far-reaching criticisms led to radical changes in the original manuscript.

J. C. L. Fish.

Stanford University, California,
March, 1915.

CONTENTS

PART I. INTRODUCTORY

CHAPTER I

THE PROBLEM OF ECONOMIC SELECTION

PART II. ELEMENTS OF THE PROBLEM OF ECONOMIC SELECTION

CHAPTER II

INTEREST

A. Simple Interest

B. Compound Interest

CHAPTER III

Sinking Funds

CHAPTER IV

First Cost

CHAPTER V

Salvage Value

A. Definitions

B. Salvage Value Fixed by Buyer and Seller

C. Salvage Value Fixed by Third Party

CHAPTER VI

Elements of Yearly Cost of Service

A. Definitions and a Schedule

B. Amortization

C. Interest and Other Fixed Charges

D. Operation

E. Maintenance

F. Outstanding data

CHAPTER VII

ESTIMATING

A. On the Practice of Estimating

B. Effect of Laws of Error on Estimates

C. On Estimating Quantities of Some Particular Types

PART III. SOLUTION OF THE PROBLEM OF ECONOMIC SELECTION

CHAPTER VIII

Basis of Economic Comparison

CHAPTER IX

Procedure for Economic Selection

CHAPTER X

Examples of Economic Selection

PART IV. BIBLIOGRAPHY AND DEPRECIATION AND LIFE TABLES

PART V. TABLES

TABLES OF FORMULAS

TABLES OF VALUES

ENGINEERING ECONOMICS

PART I

INTRODUCTORY

CHAPTER I

THE PROBLEM OF ECONOMIC SELECTION

This chapter defines "economic selection" and "structure;" shows that the problem of economic selection is the ever recurring problem in engineering; states who should be conversant with the principles of economic selection; and indicates the knowledge necessary for its solution.

1. Economic Selection.—Economic selection is choice based solely on long-run least cost. A wooden sidewalk may cost less to build than one of concrete, and yet not cost less in the long run, for the wooden sidewalk will require repairing and re-building much oftener than the concrete. To ascertain which sidewalk will cost the less "in the long run" is to solve a problem in economic selection. So, to determine which promises to be the more economical, to build the railroad around or over the hill at a low first cost and with a high annual operating cost, or through the hill at a high first cost and low operating cost, is a problem in economic selection. Other examples of economic selection will be found in Chapter X.

2. Structure.—A means of accomplishing a service consists of two parts: first, some instrument which, operating, will perform the service; and, second, the operation whereby the instrument performs the service. The instrument may be a building, a machine, a vehicle, a tool, or any part of one of such, or a plot of ground, an excavation or an embankment, a person, or a domestic animal, or any combination of the foregoing taken singly or in numbers. In this book the word "structure" will be used with the significance that is here given to the word "instrument."

Every complex structure is composed of parts each of which is itself a structure; and as the complex structure is selected and bought or rented, or selected, designed, and constructed, and operated, to perform a stated service, so each component structure is selected and bought or rented, or selected, designed, and constructed, and operated, to perform some stated service in furtherance of the service performed by the complex structure.

3. Occurrence of the Problem of Economic Selection.—The frequency of the occurrence of the problem of economic selection may be illustrated by an example taken from transportation service. Let us say that transportation service is required between two towns. Among the structures which can be made to perform this service we have the trail, highway, canal, railroad, etc. At this point is met the problem of making the economic selection from the foregoing possible structures. Let us assume that the railroad is determined to be the most economical structure for the stated service. The next step is to choose the most economical route for the railroad between the two towns. Assume the choice made, and the center line of the railroad staked out. After the railroad is located there arise the problems of selecting the most economical structures for its major parts. For example, the service to be performed by one part of the railroad is to support the track as it passes over a valley and stream therein. Shall this part of the railroad be a trestle or a bridge? Let us suppose that a bridge is ascertained to be the more economical. Next it is necessary to consider the bridge by itself. It may be built of wood, of stone or concrete, or of steel, or of a combination of any two or more of these; and it may be of any one of several types. Suppose that on making economic comparison, a steel span with concrete abutments is chosen. Next, the attention is directed to the various parts of the bridge, each in turn, and economic selection is made for each part. So the work of making economic selection goes on, even to choosing the footing course of the abutment and the paint which shall be used to cover the steel span. Let us imagine, now, that all parts of the railroad have been selected and designed in detail. After this comes the construction. There are different methods of constructing each component structure. For example, the earthwork can be executed with plows and scrapers drawn by horses, or with steam-shovels and cars, or otherwise. For each part of the earthwork one of the possible methods must be selected.

Finally, the railroad must be operated, and economic selection must be made from the different feasible ways of conducting the operation as a whole, and from the various ways of conducting each component operation.

(In the foregoing illustration of the frequency of the occurrence of the problem of economic selection we have omitted to mention the fact that before any structure can be properly selected from a number of feasible structures, the major parts and characteristics of each must be selected tentatively and in a general way for the purpose of estimating cost. The omission of descriptions of advance tentative selections does not, however, in any way impair the illustration.)

Wherever it is possible to devise two or more differing structures capable of performing a stated service, or to make two or more designs for a selected structure, or to devise two or more ways of constructing a designed structure, or to devise two or more ways of operating a completed structure to perform the stated service, the problem of economic selection arises. This is true in the case of small as in the case of large structures.

4. Who Should be Conversant with the Principles of Economic Selection.—It would appear that the principles of economic selection are an important part of the knowledge which is indispensable to owners or their executives, and to all others who hold responsible positions in connection with the choice, design, construction, or operation of structures. And it may be confidently added that by intelligently applying the principles of economic selection in his work in the engineering or industrial field, a young man will establish a strong and conspicuous claim to some position which will add to his responsibilities.

5. Knowledge Necessary for the Solution of the Problem of Economic Selection.—To read with understanding the discussion of the method of solution of the problem of economic selection (Part III) the reader must know the meanings of the terms used therein, such as "first cost," "salvage value," "amortization," etc., and have a working knowledge of interest and sinking funds. This preparatory knowledge may be obtained from Part II: Chapters II and III, and §§39, 47, 50, 71–74, 76–79, 81–84. The reader who is familiar with these topics can proceed at once to Part III.

Understanding the method of Part III and being supplied with all the pertinent numerical data ready formed for substitution

in the formulas, the computer can without difficulty make all the computations required for the economic comparison of two or more structures proposed for a stated service. But to recognize, collect, and reduce all the pertinent numerical data so that reasonable values may be assigned to the quantities represented in the formulas, and to recognize, collect, and give proper weight to all outstanding data (§§47, 84, 113), one must be conversant with practically all the topics of Part II and possessed of judgment trained by practical experience in the field in which the problem lies.

PART II

ELEMENTS OF THE PROBLEM OF ECONOMIC SELECTION

CHAPTER II

INTEREST

In this chapter formulas for "simple" and "compound" interest are derived and applied to numerical examples. The formulas for simple interest will be found collected in Table I; formulas for compound interest in Table II. Tables G, H, and J are given to aid the computer of simple interest. Table A gives compound amounts. Table B gives present worths. All tables are in Part V.

A. Simple Interest

6. Simple Interest.—It is a universal business custom for the lender of money to require the borrower to pay money for the use of the money borrowed. The money paid for the use of money borrowed or otherwise owing is called "interest." When the interest is made to be directly proportional to the time of owing, it is called "simple" interest.

7. Elements of Simple Interest Problems.—The elements of problems in simple interest are:

1. The principal.
2. The interest.
3. The time.
4. The rate.
5. The amount.

1. The principal is the sum of money, borrowed or owing, upon which interest is paid or to be paid.

2. The interest, as stated in §6, is the money paid for the use of the principal. It is the custom to arrange that interest shall be paid at the end of each year (half-year, or quarter-year) for the use of the principal during that year (half-year, or quarter-year). (For the case in which interest is paid in advance, see Bank Discount, §9.) The dates on which interest falls due are

called "interest dates;" and the interval between two consecutive interest dates is called an "interest period." When money is loaned at simple interest no interest is paid on overdue and unpaid interest. For example, if \$100 is borrowed at 8 per cent. interest, payable quarterly, the interest due at the end of each quarter is \$2, but if no interest is paid until the end of the year, the interest then due will be $4 \times \$2 = \$8$.

A common custom in computing interest is to assume that each month contains 30 days and that each year contains 360 days. Nevertheless, "the courts invariably rule that the year to be used in all interest calculations is the period of 365 days."[1]

3. The time is that during which the principal is owing by the debtor to the creditor, and may be expressed in days, in months, or in years, or in all these terms.

4. The rate is defined as the ratio between the simple interest on the principal for one year and the principal. For example, if \$16 is paid for the use of \$200 for one year, the rate is 16/200 or 0.08. As the rate is usually expressed in hundredths, it is customary to speak of the rate as so many "per cent." Thus in the foregoing example, the rate of 0.08 would usually be referred to as a "rate of 8 per cent."

5. The amount is the sum of principal and interest.

8. Formulas for Simple Interest Problems.—Formulas for the solution of problems in simple interest can be written directly from the definitions given in the preceding sections.

Let
P = the principal;
i = the rate;
I = the whole interest for n years;
n = the time in years;
S = the amount.

Then

$$I = Pin \tag{1}$$
$$S = P + Pin = P(1 + in) \tag{2}$$
$$P = I/(in) \tag{3}$$
$$P = S/(1 + in) \tag{4}$$
$$n = I/(Pi) \tag{5}$$

There are numerous published tables which give by inspection the values of I for different values of P, i, and n. See Tables H and J in Part V.

[1] Watson, James C., "Interest, Discount, and Investment Tables," Ann Arbor, Mich., George Wahr.

Example.—A note for \$280.50, bearing 6 per cent. simple interest, is dated Sept. 22, 1905, and falls due June 17, 1907. What is the total simple interest on the note for the full time?

Solution on the basis of a 365-day year:

Sept. 22, 1905 to Sept. 22, 1906	1 year
Sept. 22 to 30	8 days
October	31 days
November	30 days
December	31 days
January	31 days
February	28 days
March	31 days
April	30 days
May	31 days
June	17 days
Total time	1 year 268 days

Therefore $n = 1\text{–}268/365,$

and the total interest is

$$\begin{aligned} I &= Pin \\ &= 280.50 \times 0.06 \times 1\text{–}268/365 \\ &= \$29.19. \quad \textit{Answer.} \quad \text{(See p. 199.)} \end{aligned}$$

Solution on the basis of a 360-day year:

June 17, 1907 =	1907	6	17
Sept. 22, 1905 =	1905	9	22

Time = 1 year 8 months 25 days = 1 year 265 days.

Therefore $n = 1\text{–}265/360,$

and the total interest is

$$\begin{aligned} I &= Pin \\ &= 280.50 \times 0.06 \times 1\text{–}265/360 \\ &= \$29.22. \quad \textit{Answer.} \end{aligned}$$

This example will be found worked on p. 203 by the use of Tables G and J.

9. Bank Discount.—It is a banking custom to collect interest in advance on money loaned to customers. For example: A applies to a bank for a loan of (nominally) \$100 for 90 days, and signs a 90-day note in favor of the bank for \$100 with interest

at 8 per cent. In exchange for the note the bank delivers to A not \$100 but \$98, which is \$100 less the interest ($100 \times 0.08 \times 90/360 = \$2$) computed on the face value of the note. By this custom the bank gains and the customer loses 90 days' use of the interest in the example given.

The interest when thus paid in advance is termed "bank discount." For True Discount see §19.

B. Compound Interest

10. Compound Interest.—In simple interest, as we have seen, interest is computed only on the principal. In compound interest, interest is computed not only on the principal but also on all unpaid interest which is overdue. For example, let the principal be \$1, the rate 8 per cent., and the interest period three months. If the interest is paid at the end of each quarter each payment is $0.08/4 = \$0.02$. But if no interest is paid till the end of the year

Time		0	3 mo.	6 mo.	9 mo.	1 yr.
Quarters		1st	2nd	3rd	4th	
Simple Interest	Principal	\$1.00	\$1.00	\$1.00	\$1.00	\$1.00
	Simple Interest	.02	.02	.02	.02	.08 Total
Compound Interest	Principal	\$1.00	\$1.02	\$1.0404	\$1.061208	\$1.08243216
	Interest Earned during Quarter	.02	.0204	.020808	.02122416	.08243216 Total Int.

Fig. 1.—Simple and compound interest contrasted.

the interest due at the end of the first quarter (namely, \$0.02) is added to the principal, \$1, forming a new principal, \$1.02. The new principal, \$1.02, draws interest for the second quarter, and the interest due at the end of the second quarter is therefore $1.02 \times 0.08/4 = \$0.0204$. This interest, \$0.0204, is added to the principal \$1.02, making a newer principal, \$1.0404, which draws interest during the third quarter; and so on. The difference between compound and simple interest may be made clearer by Fig. 1.

It will be observed that with compound interest the principal on which interest is computed for each interest period is the sum formed by adding together the principal of the preceding period and the interest earned by that principal during the preceding period.

11. Elements of Compound Interest Problems.—The elements of problems in compound interest are:

1. The principal.
2. The nominal rate (usually called simply "rate").
3. The effective rate.
4. The number of interest periods in one year.
5. The time.
6. The amount.
7. The compound interest.
8. The present worth.

Formulas will now be derived whereby each of the elements can be computed when some other elements are given.

12. Amount.—The amount is the sum of the principal and all the interest.

Let P = the (original) principal;
i = the rate;
h = number of interest periods in one year;
n = the time, in years;
S = the amount (at the end of n years).

During the first interest period the (original) principal, P, earns simple interest $= Pi/h$, and amounts at the end of the period to $P(1 + i/h)$. During the second interest period the new principal, $P(1 + i/h)$, earns simple interest $= [P(1 + i/h)]i/h$, and amounts to $P(1 + i/h) + [P(1 + i/h)]i/h = [P(1 + i/h)]\ (1 + i/h) = P(1 + i/h)^2$. During the third interest period the principal, $P(1 + i/h)^2$, earns simple interest $= P(1 + i/h)^2 i/h$, and amounts to $P(1 + i/h)^2 + P(1 + i/h)^2 i/h = [P(1 + i/h)^2]\ (1 + i/h) = P(1 + i/h)^3$; and so on. It is observed that the amount for any number of interest periods is the product of P by $(1 + i/h)$ raised to a power of which the exponent is the number of interest periods. As there are hn interest periods in n years the amount of P for n years is

$$S = P(1 + i/h)^{hn} \tag{6}$$

Example.—What is the amount of $10 for 13 years with interest at 5 per cent. compounded semi-annually? The amount is

$$\begin{aligned} S &= P(1 + i/h)^{hn} \\ &= 10(1 + 0.05/2)^{2\times 13} \\ &= 10 \times 1.025^{26}. \end{aligned}$$

The value of 1.025^{26} is most conveniently found (when adequate interest tables are not at hand) by the use of logarithms. The logarithm of x^m is m times the logarithm of x.

$$\begin{array}{lll} \log 1.025\ldots\ldots\ldots\ldots & = 0.01072 & \\ \text{which multiplied by.} & \quad\quad 26 & \\ \text{gives}\ldots\ldots\ldots\ldots\ldots & 0.27872 & = \log 1.025^{26}. \end{array}$$

Therefore $1.025^{26} = \text{antilog } 0.27872 = 1.900$, and

$$S = 10 \times 1.900 = \$19. \quad \textit{Answer.}$$

(The value of 1.025^{26} can be taken directly from Table A. In Table A we find, opposite "26" and under "2-1/2 per cent.," 1.900.) See Table A for examples solved by the use of compound interest tables.

13. Interest.—The compound interest is the sum by which the amount exceeds the principal.

Let P = the principal;
i = the rate;
h = number of interest periods in one year;
n = the time, in years;
S = the amount;
I = the interest.

Then
$$I = S - P. \tag{7}$$

Replacing S by its value in Eq. 6, we have

$$I = P[(1+i/h)^{hn} - 1]. \tag{8}$$

Example.—What is the interest on \$10 for five years at 8 per cent. compounded quarterly? The interest is

$$\begin{aligned} I &= 10[(1 + 0.08/4)^{4\times 5} - 1] \\ &= 10(1.02^{20} - 1) \\ &= 10(1.486 - 1) \\ &= \$4.86. \quad \textit{Answer.} \end{aligned}$$

$$\begin{array}{r|r} 1.02 & 0.00860 \\ & 20 \\ \hline 1.486 & 0.17200 \end{array}$$

(The value of 1.02^{20} can be taken from Table A. Opposite "20" and under "2 per cent." of Table A we find 1.486 as the value of 1.02^{20}.)

14. Present Worth.—The present worth of S dollars due n years hence is equal to that principal, P, which at compound interest will amount to S dollars in n years.

Let P = the present worth;
i = the rate;
h = number of interest periods in one year;
n = the time, in years;
S = the sum due n years hence.

The relation between the given sum, S, and its present worth, P, is the same as the relation between the amount, S, and the corresponding principal, and the latter relation is expressed in Eq. 6 which is

$$S = P(1 + i/h)^{hn}.$$

Solving this equation for P we find that the present worth of S due n years hence is

$$P = S\frac{1}{(1 + i/h)^{hn}}. \tag{9}$$

Example.—What is the present worth of \$10 due eight years hence if interest is 6 per cent. compounded annually? Here $S = 10$, $h = 1$, $i = 0.06$, $n = 8$, $i/h = 0.06$, and $hn = 8$; so that the present worth is

$$\begin{aligned} P &= 10\frac{1}{1.06^8} \\ &= 10 \times 0.627 \\ &= \$6.27. \quad \textit{Answer.} \end{aligned}$$

1.06	0.02531
	8
	0.20248
1	0.00000
0.627	9.79752

(We can take the value of 1.06^8 directly from Table A. Thus opposite "8" and under "6 per cent." of Table A, we find 1.594 which is the value of 1.06^8. Also the value of $1/1.06^8$ can be taken directly from Table B. Under "6 per cent." and opposite "8" we find 0.6274 which is the value of $1/1.06^8$.)

15. Rate.—The rate of interest is the ratio of the simple interest earned in one year by the principal, to the principal. That which is here denominated "rate" is sometimes called "nominal rate." A formula which gives the rate in terms of the principal and amount may be obtained by solving Eq. 6 for i. Eq. 6 is

$$S = P(1 + i/h)^{hn}$$

where P = the principal;
i = the rate;
h = number of interest periods in one year;
n = the time, in years;
S = the amount.

Dividing both sides of Eq. 6 by P, and then taking the hnth root of both sides, we have

$$(S/P)^{1/hn} = 1 + i/h,$$

which after transposition and clearing of fractions gives us as the required rate,

$$i = h[(S/P)^{1/hn} - 1]. \qquad (10)$$

Example.—If \$100 amounts to \$121.90 in five years with interest compounding semi-annually, what is the rate?

The rate is

$$\begin{aligned} i &= 2[(121.90/100)^{1/(2\times 5)} - 1] \\ &= 2(1.219^{1/10} - 1) \qquad\qquad 1.219 \quad 0.08600 \quad 10 \\ &= 2(1.02 - 1) \qquad\qquad\quad 1.02 \quad 0.00860 \\ &= 0.04; \text{ or 4 per cent.} \quad \textit{Answer.} \end{aligned}$$

16. Time.—The number of years required for a given principal to amount to a given sum under given interest conditions, can be found from Eq. 6 which is

$$S = P(1 + i/h)^{hn},$$

where P = the principal;
i = the rate;
h = number of interest periods in one year;
n = the time, in years;
S = the amount.

Dividing both sides of Eq. 6 by P and passing to logarithms, we obtain

$$\log (S/P) = hn \log (1 + i/h), \qquad (11)$$

and finally the time in years,

$$n = \frac{\log (S/P)}{h \log (1 + i/h)}. \qquad (12)$$

(This equation gives precise results only when hn is a whole number. The method of obtaining the precise result when hn is not a whole number is shown in the following example.)

Example.—With interest at 5 per cent. compounded semi-annually, how long will it take for \$10 to amount to \$20? The number of years required is

$$\begin{aligned} n &= \frac{\log (20/10)}{2 \log (1 + 0.05/2)} = \frac{\log 2}{2 \log 1.025} \\ &= \frac{0.30103}{2 \times 0.01072} \\ &= 14.04 \text{ years, or 14 years, 14+ days.} \quad \textit{Answer.} \end{aligned}$$

This answer is not precise because the number of interest periods is $2 \times 14.04 = 28.08$ which is not a whole number. The precise time is found from the approximate time in the following manner. The compound amount of \$10 for 28 interest periods is $10(1.025)^{28} = 10 \times 1.996 = \$19.96$. (1.996 is found opposite "28" and under "2–1/2 per cent." in Table A.) It remains to find how long \$19.96 must run at simple interest to amount to \$20; or, in other words, how long it will take \$19.96 to earn $\$20 - 19.96 = \$0.04$ in simple interest. By Eq. 5 the time in years required for \$19.96 to earn \$0.04 is $0.04/(19.96 \times 0.05) = 0.04$ year, or $360 \times 0.04 = 14.5$ days. Thus the precise time required for \$10 to amount to \$20, is 14 years 14.5 days.

17. Effective Rate.—In order to avoid ambiguity the term "nominal rate" will be used in this section for that which elsewhere in this book is called "rate."

It will be remembered (§7 (4)) that

$$\begin{matrix}\text{nominal}\\ \text{rate}\end{matrix} = \frac{\text{simple interest on the principal for one year}}{\text{the principal}} \tag{13}$$

The following equation will serve to define "effective rate:"

$$\begin{matrix}\text{effective}\\ \text{rate}\end{matrix} = \frac{\text{compound interest on the principal for one year}}{\text{the principal}} \tag{14}$$

Thus the principal multiplied by the effective rate gives the compound interest on the principal for one year.

Let P = the principal;
i = the nominal rate;
e = the effective rate;
h = number of interest periods in one year.

Then
$$Pe = P[(1 + i/h)^h - 1],$$

and dividing both sides of the equation by P, we find the effective rate

$$e = (1 + i/h)^h - 1. \tag{15}$$

Example.—What is the effective rate corresponding to 8 per cent. compounded quarterly? The effective rate is

$$e = (1 + 0.08/4)^4 - 1 = 0.08243216, \text{ or } 8.243216 \text{ per cent.}$$

Answer.

For the nominal rate which, compounding h times a year, is

the equivalent of a given effective rate, e, compounding once a year, it is only necessary to solve Eq. 15 for i; thus, transposing, we have

$$(1 + i/h)^h = 1 + e; \tag{16}$$

and taking the hth root of both sides, there results

$$1 + i/h = (1 + e)^{1/h},$$

whence we obtain the nominal rate

$$i = h[(1 + e)^{1/h} - 1]. \tag{17}$$

Example.—What nominal rate compounded quarterly corresponds to the effective rate, 8.2 per cent.? The corresponding nominal rate is

$$\begin{aligned} i &= 4(1.082^{1/4} - 1) \\ &= 4(1.02 - 1) \\ &= 0.08, \text{ or 8 per cent.} \quad \textit{Answer.} \end{aligned}$$

1.082	0.03423	4
1.02	0.00856	

(*Note.*—Evidently this answer is not exact, for we see from the example under Eq. 15 that the nominal rate is 8 per cent. for an effective rate 8.243216 per cent.)

18. When One of the Given Elements is \$1.—In the foregoing sections the given principal or given amount has been taken to be any sum whatever. In practice it is often convenient to use formulas based upon the assumption that the given principal or the given amount is unity, and this convenience is increased by using some special notation.

Let p = present worth of \$1 due one year hence;
p_n = present worth of \$1 due n years hence;
s = amount of \$1 for one year;
s_n = amount of \$1 for n years;
i = the rate;
I' = the interest on \$1 for n years;
h = number of interest periods in one year;
n = the time, in years.

The amount of \$1 for one year is (Eq. 6),

$$s = (1 + i/h)^h, \tag{18}$$

and the amount of \$1 for n years is (Eqs. 6 and 18),

$$\begin{aligned} s_n &= (1 + i/h)^{hn} && (18a) \\ &= s^n. && (18b) \end{aligned}$$

(s_n can be taken directly from Table A when $h = 1$. See examples accompanying Table A.)

The interest on \$1 for n years is (Eqs. 7, 18a, and 18b),

$$I' = s_n - 1 = (1 + i/h)^{hn} - 1 \tag{19}$$
$$= s^n - 1. \tag{19a}$$

The present worth of \$1 due one year hence is (Eq. 9),

$$p = \frac{1}{(1+i/h)^h} \tag{20}$$
$$= 1/s \tag{20a}$$

and the present worth of \$1 due n years hence is (Eqs. 9 and 20),

$$p_n = \frac{1}{(1+i/h)^{hn}} \tag{20b}$$
$$= 1/s_n \tag{20c}$$
$$= p^n. \tag{20d}$$

(p_n can be taken directly from Table B when $h = 1$. See examples accompanying Table B.)

The rate required to make \$1 amount to a given sum, s_n, in n years, is (Eq. 10),

$$i = h(s_n^{1/hn} - 1). \tag{21}$$

The time, n (in years), required for \$1 to amount to a given sum, s_n, is (Eq. 12),

$$n = \frac{\log s_n}{h \log (1 + i/h)}. \tag{22}$$

19. True Discount.—"True discount" is a term sometimes applied to the present worth of the interest computed on the face value of a note.

Let P = the principal (face value of the note);
i = the rate;
h = number interest periods in one year;
n = the time, in years;
I = the interest for n years;
T = the true discount.

From Eq. 9 and the definition above, the true discount is

$$T = \frac{I}{(1 + i/h)^{hn}}. \tag{23}$$

From Eq. 8 we have $I = P[(1 + i/h)^{hn} - 1]$. Putting this value of I in Eq. 23, we have

$$T = \frac{P[(1 + i/h)^{hn} - 1]}{(1 + i/h)^{hn}} \tag{23a}$$

or, from Eqs. 18a, 20b, and 23a,

$$T = P(s_n - 1)p_n. \tag{23b}$$

s_n can be taken from Table A, and p_n can be taken from Table B, when $h = 1$.

For Bank Discount, see §9.

Example.—What is the true discount on a \$100 note for two years with interest at 8 per cent. compounded annually?

Solution by Eq. 23b, using Tables A and B: opposite "2" and under "8 per cent." in Table A we find $s_n = 1.166$; and opposite "2" and under "8 per cent." in Table B we find $p_n = 0.8573$. Therefore the true discount is

$$T = 100(1.166 - 1)0.8573 = \$14.23. \quad \textit{Answer.}$$

CHAPTER III

SINKING FUNDS

This chapter describes "sinking funds," "annuities," "present justifiable expenditure," "capitalized value," and "installments," and shows that problems concerning them are all mathematically the same. Formulas are derived, and all except Eqs. 39 to 43 (both inclusive) are collected in Tables III, IV, V, and VI. The numerical Tables C, D, E, and F are for use in solving problems under this chapter.

A. Introductory

20. Sinking Funds.—It may be well to precede the definition of a sinking fund by a few concrete examples.

A owes B \$1000 to be paid 10 years hence. A pays B the interest due each year; and, in order to provide gradually for paying the principal, places \$100 at the end of each year in a strong-box which is otherwise empty. At the end of the tenth year, immediately after putting in the tenth deposit of \$100, A hands the contents of the box to B, and thereby discharges in full his obligation to B. The growing fund in the strong-box has all the characteristics of a sinking fund except that the deposits do not draw interest.

If, in the foregoing example, A had deposited each \$100 in a savings bank that paid 4 per cent. interest compounded annually, instead of dropping it in the strong-box, he would have had at the end of the tenth year a credit of 10×100 *plus* all compound interest earned by all the deposits; and the interest would have been over and above the requirements of the obligation. In order that the compound amount of the ten deposits might be only \$1000 A ought to make the deposit in the bank somewhat less than \$100 at the end of each year. Had A chosen the bank instead of the strong-box, the increasing fund comprising deposits and interest would have been a "sinking fund." To calculate the deposit required to be put in the savings bank under the conditions above is a typical sinking-fund problem.

Suppose that C buys a machine for \$2000. He estimates that the machine will last 12 years. In order to provide for money to

buy a new machine at the end of the 12 years, he sets aside at the end of each year a sum taken from the earnings of the machine—a uniform sum which he deposits in a savings bank to draw compound interest. He thus establishes a sinking fund for the renewal of the machine.

To raise money for some purpose a corporation sells ten 20-year bonds of the par value of $1000 each, bearing interest at 5 per cent. per annum. To provide for the redemption of the bonds—the amortization (§73) of the obligation—at maturity, the corporation sets aside at the end of each of the 20 years a uniform sum and invests the same at 8 per cent. compounding semi-annually. That the investments may amount to $20,000 at the end of the twentieth year, what must be the annual sum so invested? The investments here constitute a sinking fund, and the question constitutes a sinking-fund problem.

Definition.—A sinking fund is a fund built up during a period of time to provide a given sum of money at the end of that period, by making at regular intervals uniform deposits which draw compound interest. The interest rate, deposit interval, and size of deposit are so chosen as to make the sum of deposits plus total compound interest thereon equal to the given sum of money at the end of the period of time.

21. Growth of Sinking Fund.—The growth of a sinking fund may be shown by the following simple example. We deposit $1 in a savings bank at the end of each year during a period of four years, to draw interest at 4 per cent. compounded annually. After the first $1 is deposited we have at the bank a growing fund of credit which may be called a sinking fund. The $1 deposited at the end of the first year draws compound interest for three years and amounts to $1.124864; the $1 deposited at the end of the second year draws interest for two years and amounts to $1.0816; the $1 deposited at the end of the third year draws interest for one year and amounts to $1.04; and the $1 deposited at the end of the fourth, and last, year draws interest for no time and therefore amounts to $1. The amount of the sinking fund at the end of the fourth year is evidently the sum of the four individual amounts, or $4.246464.

The building process used in the foregoing illustration of the growth of a sinking fund might be used to find the amount of a sinking fund in which $1 is deposited at the end of each year for any number of years, as 10, say. Assume that the deposits

draw interest at 4 per cent. compounded annually. Taking the amount of $1 for each time from Table A, we have:

1st deposit runs 9 years and amounts to	$1 423
2d deposit runs 8 years and amounts to	1 369
3d deposit runs 7 years and amounts to	1 316
4th deposit runs 6 years and amounts to	1 265
5th deposit runs 5 years and amounts to	1 217
6th deposit runs 4 years and amounts to	1 170
7th deposit runs 3 years and amounts to	1 125
8th deposit runs 2 years and amounts to	1 082
9th deposit runs 1 year and amounts to	1 040
10th deposit runs 0 years and amounts to	1 000
Amount of 10-year sinking fund =	$12 007

But fortunately the same result can be reached by a process much shorter than this, as is shown farther on.

22. Elements of a Sinking Fund.—The elements of a sinking fund are:

1. The deposit.
2. The deposit interval.
3. The life.
4. The rate.
5. The number of interest periods in one year.
6. The amount.
7. The present worth.

In any sinking-fund problem some of these elements are given and one or more of the others are sought.

B. Formulas for Sinking Funds When Deposit Is Made at the End of Each Deposit Interval and Deposit Interval = or > Interest Period

23. Amount.—The amount of a sinking fund is the sum of the amounts of the deposits at the end of the life of the fund.

Let Z = the amount;
D = the deposit;
k = number of years in one deposit interval (k is a multiple of $1/h$);
n = the life in years (n is a multiple of k);
s = amount of $1 for one year;
$= (1 + i/h)^h$;
i = the rate;
h = number of interest periods in one year.

Looking at Fig. 2, which shows the relations between the elements of a sinking fund, we see that the first deposit, D, runs $n-k$ years and amounts to Ds^{n-k}. The second deposit, D, runs $n-2k$ years and amounts to Ds^{n-2k}. And so on down to the last deposit, D, which runs no time at all and therefore amounts to D. The amount of the sinking fund is the sum of all these amounts written in the right-hand column of Fig. 2; therefore

$$Z = D(1 + s^k + s^{2k} + \ . \ . \ . \ + s^{n-2k} + s^{n-k}) \tag{24}$$

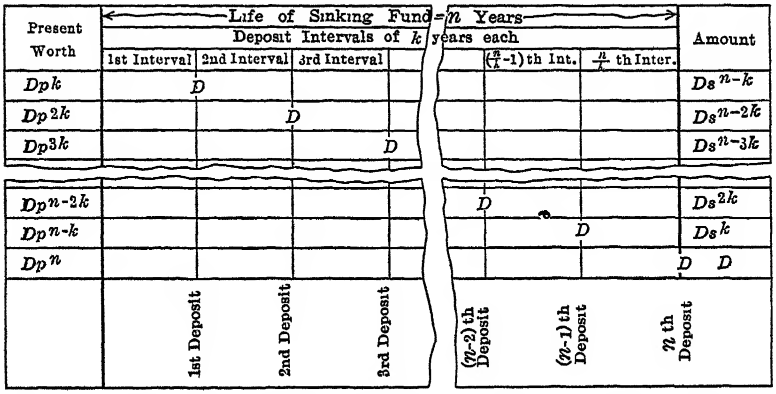

FIG. 2.—Sinking fund—general case showing relations between elements.

The terms within the parentheses constitute a geometric series. From algebra we learn that the

$$\text{sum of a geometric series} = \frac{\text{last term} \times \text{ratio} - \text{first term}}{\text{ratio} - 1}. \tag{25}$$

In the series above, the last term is s^{n-k}, the ratio is s^k, and the first term is 1. Therefore

$$1 + s^k + s^{2k} + \ . \ . \ . \ + s^{n-2k} + s^{n-k} = \frac{s^{n-k}\, s^k - 1}{s^k - 1} = \frac{s^n - 1}{s^k - 1}. \tag{26}$$

By substituting from Eq. 26 in Eq. 24, we find the amount to be

$$Z = D\,\frac{s^n - 1}{s^k - 1}, \tag{27}$$

or, expanding s,
$$Z = D\,\frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^{hk} - 1}. \tag{27a}$$

See the example accompanying Table IV.

24. Present Worth.—The present worth of a sinking fund is equal to the present worth of the amount of the fund, and is the sum of the present worths of the deposits.

Let Z = the amount;
W = the present worth;
D = the deposit;
k = the deposit interval, in years (k is a multiple of $1/h$);
n = the life, in years (n is a multiple of k);
s = amount of \$1 for one year;
$= (1 + i/h)^h$;
i = the rate;
h = number of interest periods in one year.

Remembering that the relation between W and Z is such that $Z = Ws^n$ (Eqs. 6 and 18a), substituting Ws^n for Z in Eq. 27, and solving for W, we obtain the present worth

$$W = D\frac{s^n - 1}{s^k - 1} \cdot \frac{1}{s^n}, \tag{28}$$

or, expanding s,

$$W = D\frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^{hk} - 1} \cdot \frac{1}{(1 + i/h)^{hn}}. \tag{28a}$$

See example opposite Table IV, involving a similar formula.

25. Deposit.—By solving Eqs. 27 and 27a for D we obtain the value of the deposit in terms of the amount; and by solving Eqs. 28 and 28a for D we obtain the value of the deposit in terms of the present worth.

Let D = the deposit;
Z = the amount;
W = the present worth;
k = the deposit interval, in years (k is a multiple of $1/h$);
n = the life, in years (n is a multiple of k);
s = the amount of \$1 for one year;
$= (1 + i/h)^h$;
i = the rate;
h = number of interest periods in one year.

From the solutions referred to above

$$D = Z\frac{s^k - 1}{s^n - 1} \tag{29}$$

$$= Z\frac{(1 + i/h)^{hk} - 1}{(1 + i/h)^{hn} - 1} \tag{29a}$$

$$= W\frac{(s^k - 1)s^n}{(s^n - 1)} \tag{30}$$

$$= W\frac{[(1 + i/h)^{hk} - 1](1 + i/h)^{hn}}{(1 + i/h)^{hn} - 1}. \tag{30a}$$

All the formulas of Table IV, which include the foregoing, have been derived from Eqs. 27 and 28, but the details of the work are omitted. See the numerical example accompanying Table IV.

C. Formulas for Sinking Funds When Deposit Is Made at the End of Each Year and $1 is One of the Given Elements

26. Amount and Present Worth when the End-of-year Deposit is $1.—For this case k and D of Eqs. 27, 27a, 28, and 28a become unity. Fig. 3.

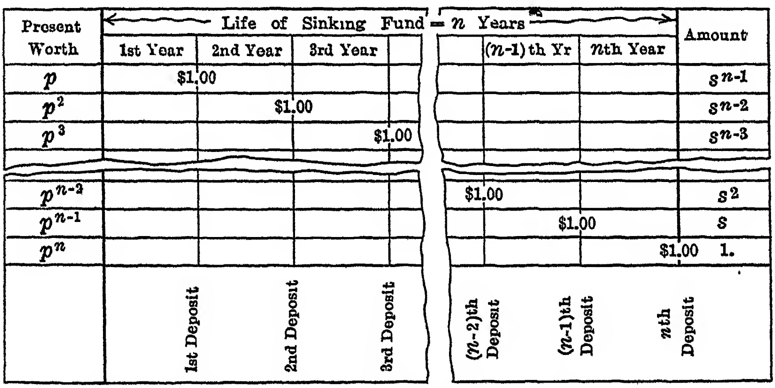

FIG. 3.—Sinking fund in which end-of-year deposit is $1.

Let $\$1$ = the deposit;
n = the life, in years (n is a multiple of $1/h$);
z_n = the amount;
w_n = the present worth;
s = the amount of $1 for one year;
$= (1 + i/h)^h$;
i = the rate;
h = number of interest periods in one year.

Under the assumptions and notation above, Eq. 27 becomes

$$z_n = \frac{s^n - 1}{s - 1}, \tag{31}$$

Eq. 27a becomes $$z_n = \frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^h - 1}. \tag{31a}$$

Table C gives values of z_n for various values of n and i for cases in which $h = 1$.

Eq. 28 becomes $$w_n = \frac{s^n - 1}{s - 1} \cdot \frac{1}{s^n}, \tag{32}$$
and Eq. 28a becomes

$$w_n = \frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^h - 1} \cdot \frac{1}{(1 + i/h)^{hn}}. \tag{32a}$$

Table D gives values of w_n for various values of n and i for cases in which $h = 1$.

27. End-of-year Deposit Required to Make Sinking Fund Amount to $1.—Here k and Z of Eqs. 29 and 29a become unity.

Let $\$1$ = the amount;
d_n = the deposit;
n = the life, in years (n is a multiple of $1/h$);
s = the amount of $1 for one year;
= $(1 + i/h)^h$;
i = the rate;
h = number of interest periods in one year.

Then Eq. 29 reduces to
$$d_n = \frac{s - 1}{s^n - 1}, \tag{33}$$
and Eq. 29a reduces to
$$d_n = \frac{(1 + i/h)^h - 1}{(1 + i/h)^{hn} - 1}. \tag{33a}$$

Table E gives values of d_n for various values of n and i for cases in which $h = 1$.

28. End-of-year Deposit in Sinking Fund of which the Present Worth is $1.—In this case k and W of Eqs. 30 and 30a become unity.

Let $\$1$ = the present worth;
a_n = the deposit;
n = the life, in years (n is a multiple of $1/h$);
s = the amount of $1 for one year;
= $(1 + i/h)^h$;
i = the rate;
h = number of interest periods in one year.

For this special case, D, of Eqs. 30 and 30a, becomes a_n, and Eq. 30 becomes

$$a_n = \frac{(s - 1)s^n}{s^n - 1}, \tag{34}$$

and Eq. 30a becomes

$$a_n = \frac{[(1 + i/h)^h - 1](1 + i/h)^{hn}}{(1 + i/h)^{hn} - 1}. \tag{34a}$$

Table F gives values of a_n for various values of n and i for cases in which $h = 1$. See examples accompanying Table F.

The foregoing will be found with additional formulas in Table III.

D. Formulas for Sinking Funds When the Deposit Is Made at the Beginning of Each Deposit Interval and the Deposit Interval = or > the Interest Period

29. Sinking Funds in which Deposit is Made at the Beginning of Each Deposit Interval.—By the processes used in the preceding pages the formulas which appear in Table V have been derived for the cases in which the deposit, D, is made at the *beginning of each k-year interval* of the life of the sinking fund. The formulas assume that the deposit interval is equal to one or more whole interest periods, and that the life is one or more whole deposit intervals. See the example accompanying Table V.

30. Sinking Funds in which Deposit is Made at the Beginning of Each Deposit Interval and at the End of the Last Deposit Interval.—In Table VI will be found formulas for cases in which the deposit, D, is made at the *beginning of each* deposit interval and at the *end of the last* deposit interval of the life of the fund. The formulas assume that the life is equal to one or more whole deposit intervals, and that a deposit interval is equal to one or more whole interest periods. These formulas have been derived in the same way as those of Table IV. See example opposite Table VI.

E. Formulas for Sinking Funds When the Deposit Is Made at the End of Each Deposit Interval and the Interval is Less than the Interest Period

The foregoing formulas will not serve when the deposit interval is less than the interest period; as, for example, in the case of computing, on the basis of quarterly compounding, the monthly payment over a given time, which shall be the equivalent of a given cash payment. Formulas for this case will now be derived.

31. Amount.—The amount of the sinking fund is the sum of the amounts of the deposits. It should be observed that those deposits which are made between two consecutive interest dates draw *simple* interest until the arrival of the second of the two dates.

Let D = the deposit;
i = the rate;
h = number of interest periods in one year;
n = the life, in years (n is a multiple of $1/h$);
t = number of deposit intervals in one interest period;
G = the amount, at simple interest, at the end of each interest period, of the t deposits made during that period;
Z = the amount of the sinking fund.

The first step is to find G.

The first deposit, D, runs $t - 1$ deposit intervals at simple interest and amounts to $D + D\frac{i}{ht}(t - 1)$; the second deposit, D, runs $t - 2$ deposit intervals at simple interest and amounts to $D + D\frac{i}{ht}(t - 2)$; and so on. The next to the last deposit runs one interval and amounts to $D + D\frac{i}{ht}$, and the last runs no time and amounts to D. The sum of these amounts is G; therefore

$$G = tD + D\frac{i}{ht}[1 + 2 + 3 + \quad . \quad . \quad . \quad + (t - 2) + (t - 1)]. \quad (35)$$

The series within the brackets is an arithmetic progression. From algebra we have

$$\text{sum of arithmetic progression} = \frac{\text{No. of terms} \times (\text{first term} + \text{last term})}{2}. \quad (36)$$

In the series of Eq. 35 the number of terms is $t - 1$, the first term is 1, and the last term is $t - 1$. Therefore

$$1 + 2 + 3 \quad . \quad . \quad . \quad + (t - 2) + (t - 1) = \frac{(t - 1)[1 + (t - 1)]}{2}$$
$$= \tfrac{1}{2}(t - 1)t; \quad (37)$$

and Eq. 35 reduces to

$$G = tD + D\frac{i}{ht}[\tfrac{1}{2}(t - 1)t]$$

or

$$G = D\left[t + \frac{t - 1}{2} \cdot \frac{i}{h}\right]. \quad (38)$$

Now if we substitute this value of G for the D of Eq. 27a,

$$Z = D\,\frac{(1+i/h)^{hn}-1}{(1+i/h)^{hk}-1}$$

and put

$$k = 1/h$$

that equation becomes

$$Z = D\left[t+\left(\frac{t-1}{2}\right)\frac{i}{h}\right]\frac{(1+i/h)^{hn}-1}{i/h} \tag{39}$$

where Z = the amount of the sinking fund;
D = the deposit;
t = number of deposit intervals in one interest period;
i = the rate;
h = number of interest periods in one year;
n = the life, in years (n is a multiple of $1/h$).

(It should be observed that

$$\frac{(1+i/h)^{hn}-1}{i/h}$$

is the amount of a sinking fund in which \$1 is deposited at the end of each year for hn years with interest at the rate i/h compounded annually. Therefore the value of this fraction can be taken from Table C when the values of hn and i/h come within the range of the table.)

32. Present Worth.—The present worth of the sinking fund is the present worth of the amount of the sinking fund. Therefore (Eq. 9),

$$W = Z\,\frac{1}{(1+i/h)^{hn}}. \tag{40}$$

Substituting for Z in Eq. 40 the value of Z given in Eq. 39, we have

$$W = D\left[t+\left(\frac{t-1}{2}\right)\frac{i}{h}\right]\frac{(1+i/h)^{hn}-1}{(i/h)(1+i/h)^{hn}} \tag{41}$$

where W = the present worth;
D = the deposit;
t = number of deposit intervals in one interest period;
i = the rate;
h = number of interest periods in one year;
n = the life, in years (n is a multiple of i/h).

33. Deposit.—If we solve Eq. 39 for D we obtain

$$D = Z\,\frac{i/h}{\left[t+\left(\frac{t-1}{2}\right)\frac{i}{h}\right]\left[(1+i/h)^{hn}-1\right]} \tag{42}$$

where Z is the amount and the other letters have the same meanings as in the preceding section.

By solving Eq. 41 for D, we obtain the deposit in terms of the present worth:

$$D = W \frac{(i/h)(1 + i/h)^{hn}}{\left[t + \left(\frac{t-1}{2}\right)\frac{i}{h}\right]\left[(1 + i/h)^{hn} - 1\right]} \tag{43}$$

where the notation is the same as in the preceding section.

Example.—A buys of B a lot the price of which is \$1000. B offers to let A pay cash or pay in monthly installments such as will pay B 8 per cent. compounded quarterly for the use of his money. A decides to pay in installments, so much at the end of each month for four years. What is the monthly installment? Here the principal and installment correspond respectively to the present worth and deposit of the sinking fund described above. So Eq. 43 will serve our need here. For this example $W = 1000$, $i = 0.08$, $h = 4$, $t = 3$, and $n = 4$; and the installment is

$$D = 1000 \frac{0.02 \times 1.02^{16}}{\left[3 + \frac{3-1}{2}(0.02)\right]\left(1.02^{16} - 1\right)}.$$

$1.02^{16} = 1.373$ (taken from Table A, column of "2 per cent." and opposite "16"); therefore

$$D = 1000 \frac{0.02 \times 1.373}{3.02 \times 0.373} = \$24.38. \quad \textit{Answer.}$$

F. Annuities, Present Justifiable Expenditure, Capitalized Value, and Installments

34. Annuities.—The regular yearly payment of a uniform sum of money is often termed an "annuity." The elements of annuity problems are:

1. The principal (otherwise called "investment" or "present worth," corresponding to the present worth of a sinking fund (§20)).
2. The periodic payment (otherwise called "yield," corresponding to the deposit of a sinking fund).
3. The interval between payments (corresponding to the deposit interval of a sinking fund).
4. The life, in years (corresponding to the life of a sinking fund).
5. The rate (corresponding to the rate of interest earned by deposits in a sinking fund).

6. The number of interest periods in one year (the same as for a sinking fund).

7. The amount (the same as for a sinking fund).

The mathematical relations between the elements of an annuity are the same as the relations between the corresponding elements of a sinking fund. Therefore the formulas which serve for sinking-fund problems serve also for problems in annuities. Compare Figs. 3 and 4. Four formulas which have been derived for sinking funds will be inserted here for the convenience of the reader.

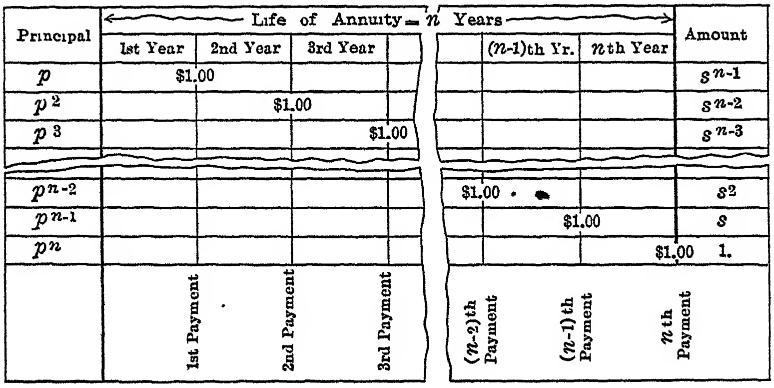

FIG 4.—Annuity of $1 at end of each year.

Let w_n = the principal which yields an annuity of $1 at the end of each year for n years;

a_n = the end-of-year annuity yielded for n years by a principal of $1;

z_n = the amount of an annuity of $1 at the end of each year for n years;

n = the life of the annuity, in years;

s = the amount of $1 for one year;

$= (1 + i/h)^h$;

i = the rate;

h = number of interest periods in one year.

The end-of-year yield from a principal of $1 is (Eq. 34),

$$a_n = \frac{(s-1)s^n}{s^n - 1}, \tag{44}$$

or (Eq. 34*a*),

$$a_n = \frac{[(1+i/h)^h - 1](1+i/h)^{hn}}{(1+i/h)^{hn} - 1}. \tag{44a}$$

(Table F gives values of a_n for various values of n and i for cases in which $h = 1$. See examples accompanying Table F.)

The principal required to yield \$1 at the end of each year is (Eq. 32),

$$w_n = \frac{s^n - 1}{(s-1)s^n}, \tag{45}$$

or (Eq. 32a),
$$w_n = \frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^h - 1} \cdot \frac{1}{(1 + i/h)^{hn}}. \tag{45a}$$

(Table D gives values of w_n for various values of n and i for cases in which $h = 1$. For examples, see Table D.)

The foregoing formulas will serve our present purpose. As has been said above, all the sinking-fund formulas are available for the solution of problems in annuities. See Tables III, IV, V, VI, and §§31–33.

Example 1.—What principal must be invested at 4 per cent. compounded annually to yield an end-of-year annuity of \$500 for 10 years?

Opposite "10" and under "4 per cent.," Table D, we find 8.111 which is the principal required to yield an annuity of \$1 per year under the given conditions. The principal sought is therefore $500 \times 8.111 = \$4055.50\pm$. *Answer.*

Example 2.—For how many years will the principal, \$10,000, invested at 5 per cent. compounded annually, yield an end-of-year annuity of \$1000? This example may be solved by the use of the following formula taken from Table IV.

$$n = \frac{\log \left[\dfrac{1}{1 - (W/D)(s^k - 1)}\right]}{\log s}.$$

Here $W = 10{,}000$, $D = 1000$, $s^k = (1 + i/h)^{hk}$, $i = 0.05$, $h = 1$, and $k = 1$. Therefore $W/D = 10$, $s^k = 1.05$, $s^k - 1 = 0.05$, and

$$n = \frac{\log \left[\dfrac{1}{1 - 10\,(0.05)}\right]}{\log 1.05} = \frac{\log 2}{\log 1.05}.$$
$$= 0.3010/0.0212$$
$$= 14.2 \text{ years.} \quad \textit{Answer.}$$

This answer being fractional shows that when the fourteenth yearly payment has been made there will remain a small balance insufficient with interest to meet a fifteenth payment of \$1000.

Example 3.—To provide an annuity for four years, $1000 is invested at 8 per cent. compounded quarterly. The annuity is to be paid in monthly installments, a fixed sum at the end of each month. What will the monthly installment be? Here the payment interval is less than the interest period, and computation must be made by Eq. 43. The solution of this example is given in the example which follows Eq. 43.

35. Present Justifiable Expenditure.—"Present justifiable expenditure" is a term sometimes used to signify present worth (§14). What is the present justifiable expenditure to avoid the payment of $100 one year hence? Evidently it is the present equivalent or present worth. What is the present justifiable expenditure to effect a saving of $50 per year for eight years? It is in this case

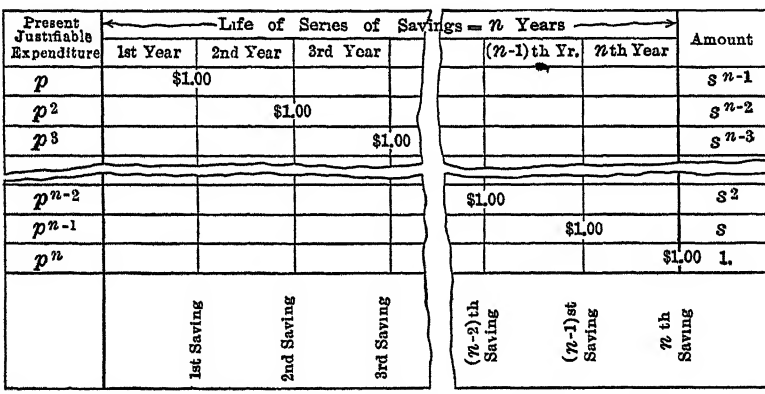

FIG. 5.—Present justifiable expenditure to save $1 at end of each year.

the sum of the present worths of the individual yearly savings. The justifiable expenditure to effect a uniform series of savings bears the same relation to the series of savings as the present worth of a sinking fund (§24) bears to the series of deposits in the sinking fund. Compare Figs. 3 and 5.

From the foregoing it is to be inferred that the formulas for present worth in Tables II, III, IV, V, and VI will serve for the calculation of present justifiable expenditure.

Example.—What is the present justifiable expenditure to avoid the payment of $100 at the end of each year for 10 years if money is worth 6 per cent. compounded annually? Opposite "10" and under "6 per cent.," Table D, we find 7.360 which is the present justifiable expenditure to avoid the payment of $1 at the end of

each year for 10 years. The answer to the example is therefore $100 \times 7.360 = \$736.00\pm$.

36. Capitalized Value.—Present worth is sometimes called "capitalized value." The "capitalized cost" of maintaining a structure is the present worth of the cost of maintaining the structure. The capitalized value of an income of $2500 per year for eight years is the present worth of $2500 per year for eight years. Hence the formulas for present worth in Tables II, III, IV, V, and VI, can be used to find capitalized cost or value.

A comparison of Figs. 3 and 6 shows that the elements of the problem of capitalized value in the case of a regular series of

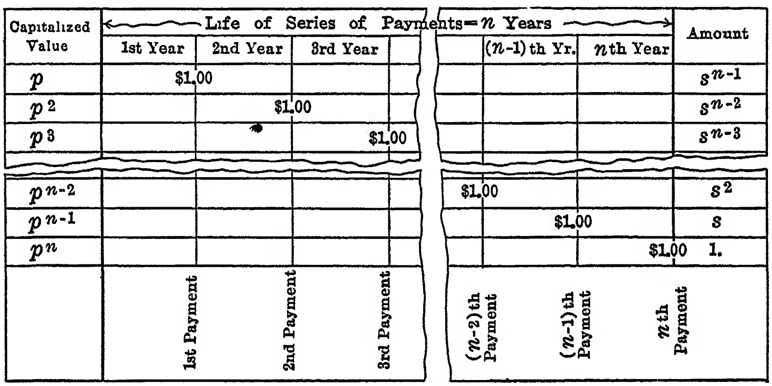

FIG. 6.—Capitalized value of $1 at end of each year.

payments bear the same relation to one another as do the elements of the sinking-fund problem. Compare also Figs. 4, 5, and 6.

Example.—What is the capitalized cost of maintenance for 10 years if the yearly cost is $100 and money is worth 6 per cent. compounded annually? The capitalized cost of maintenance is $100 \times 7.360 = \$736.00\pm$. (7.360, taken from line "10" column "6 per cent.," Table D, is the capitalized cost of maintenance at $1 per year for 10 years.)

37. Installments.—Under the "installment plan" the purchaser agrees to pay so much (some aliquot part of the purchase price) per week, month, or year, without interest, until the sum of the partial payments is equal to the purchase price, and that the title to the thing paid for shall remain with the seller until the final partial payment is made. It is to be assumed that the

seller makes the installment-plan price greater than the cash price by enough to pay him for waiting for his money.

It is plain that the series of installments corresponds to the series of deposits in a sinking fund (§22). Hence problems in installments are the same mathematically as the sinking-fund problem, and the formulas of Tables III, IV, V, and VI can be used to solve all problems in installments in which the interval between payments is equal to or greater than the interest period; and Eqs. 39 to 43 can be used when the interval between payments is less than the interest period.

See the example under Eq. 43.

CHAPTER IV

FIRST COST

This chapter gives some idea of "investigation" and "promotion;" points out that first cost is the sum of the costs of investigation, promotion, and construction (or purchase and installation); classifies the items of first cost and briefly discusses each, including contingencies and outstanding data.

38. Investigation, Promotion, and Construction (or **Purchase**). —Occasionally we see the owner of a structure casting about for something for it to do, as when he has inherited the structure, or taken it on a bad debt, or bought it at the height of a bargain fever, or when he has ceased to need the service for which he bought or built the structure. Usually, however, the idea of service to be performed precedes the creation (or purchase) of the actual structure.

Leading, directly or indirectly, from the idea of the service to the completion (or installation) of the structure created (or purchased) to perform the service there is required a series of steps each of which is a necessary preliminary to some following step.

Here is a simple example which shows the course of the steps taken to develop an idea of service into an actual structure. A is struck with the idea that a railroad between two cities would pay. He makes some investigations to ascertain the probable traffic of the road, and causes a survey to be made, and from the investigation and survey derives estimates of future revenue, first cost, and operating expenses. The estimates indicate that the road would be a paying investment. To raise the money to build the road, A lays his estimates before capitalists B, C, and D. These men are induced by A's showing to put their money into the enterprise. A, B, C, and D then form a partnership or company, which in turn constructs the road.

A similar, but more complex, example is the following: A long haul by wagon road is required to convey ore from the mining district, W, to the nearest shipping point, Z. E conceives the idea that a railroad between W and Z would be profitable. He

makes some rough calculations on cost and revenue, and obtains results which support the idea. He engages an engineer to make a reconnaissance and estimates. The engineer's estimates corroborate those of E. Having no money to carry the investigation farther, E lays the proposition before capitalists F, G, H, and J; and by means of the engineer's estimates induces them to join with him in forming a railroad company and securing a charter. The company makes a preliminary survey. The estimates based on this survey warrant the building of the road. The company issues bonds, and by means of the showing made by the preliminary survey, induces capitalists to buy the bonds at such a price and in such numbers as to bring the company a sum sufficient to build. The company then locates and builds the road.

Reduced to its lowest terms the procedure leading from the conception of the idea that a potential demand for a stated service exists and that a structure to perform the service can be created (or purchased) and operated with profit, to the completion (or installation) of the structure, is as follows:

1. *Investigation.*—The soundness of the idea is tested and confirmed by the collection and study of data.

2. *Financing.*—The idea together with data and arguments is presented to a capitalist who becomes convinced of the soundness of the idea and agrees to supply the money needful for the creation (or purchase) of the structure.

3. *Organization.*—A business organization is effected.

4. *Construction (or Purchase).*—The structure is built (or purchased).

Financing and organization may be included under the one term "promotion." Thus every step in the series falls in one or another of the main divisions of the procedure: investigation, promotion, and construction (or purchase and installation).

39. What First Cost Is.—The sum of all the expenditures made for investigation, promotion, and construction (or purchase and installation), is the "first cost" of the structure.

The number and variety of steps taken during investigation, promotion, and construction (or purchase), and the diversity of the items of expenditure under each step depend in any given case upon the extent and complexity of the structure and the relation of the structure to existing conditions.

Before entering upon the classification of expenditures it is proper to consider the two principles which follow.

First Cost of Structure Includes Outlays on Other Structures Imagined or Real.—The first cost of a given structure includes such expenditures on, and for, other structures (imagined or real) as are necessitated by selecting and building (or buying and installing) the given structure. This principle is made clear by the following examples.

1. A small waterway is required under a highway. Field observations are made and studied for the purpose of gathering data upon which the size of the required opening should be based, and the cross-sectional area of the opening is determined. Then data on pipes and culverts must be collected. Let us assume that 3 square feet is the area of the required opening. For this a 24-inch pipe will do. There are different kinds of pipe: iron, steel (plain or corrugated), terra cotta, etc. To select an economical pipe requires a knowledge of the strength, durability, availability, of the cost f.o.b., and of the cost of transportation of each variety of pipe. Also there are arch- and box-culverts of wood, brick, stone, or concrete, etc. The monetary and other advantages and disadvantages of each type and of each kind of material must be considered. Finally, let us say, a stone box-culvert is selected as being on the whole the most economical. Detailed working plans must be made for the box-culvert. If the box be built all the expense incurred in studying all the rejected structures as well as the work on the stone box-culvert itself, must be included in the first cost of the box-culvert.

2. Clearing the site of a proposed structure may involve the bodily or piece-meal removal of an existing structure. The cost of this removal is a part of the first cost of the new structure.

3. The construction of the foundation of a proposed building may necessitate strengthening the foundation of an adjacent building. The cost of such strengthening is part of the first cost of the proposed building.

4. When a highway bridge is worn out the building of its successor may call for the construction of a temporary road around the site of the bridge. For this temporary road there may be required a right of way over private lands, some grading, and a temporary bridge over the stream; and the cost of all these is a part of the first cost of the new highway bridge.

Whatever is Supplied by the Owner is Part of the First Cost of the Structure.—Whatever is necessary to any step in the creation of a structure is an item of first cost of the structure, whether it be

obtained from the owner or from an outside party; and any payment which would be properly a part of the first cost of the structure if made to a party having no interest therein, is no less a part of said first cost if made to a party having interest therein. A few examples will make this clear.

1. If an architect designs his own house, the architect's commission, computed as for a client, is a part of the first cost of the house just as truly as if the design were made by another architect.

2. Hardware is an item of the first cost of a house. If the owner of the house happen to be a hardware merchant and get the hardware from his own store the case will not be altered; and the cost should be computed on the basis of prices charged others.

3. The cost of railroad transportation on material for a new bridge is a part of the first cost of the bridge whether the bridge and the railroad are under one ownership or not.

40. Classification of Items of First Cost.—A common source of mistakes in estimates of first cost lies in the omission of items. Chance for such mistakes to occur is reduced to a minimum if the estimator have at hand a schedule of all the items for which costs must be estimated, and check off each item as it is estimated. In offices where estimates are continually made for structures of a particular class, such a schedule of items, made complete by discovering and adding an overlooked item from time to time, may be printed and offer all the advantage of a printed laundry list. The helpfulness of such a detailed schedule will be appreciated on reading page 43 of Gillette's "Cost Data"[1] and Ford's "Valuation of Intangible Street Railway Property."[2]

It would be appropriate to set forth here a detailed schedule so comprehensive as to be applicable to every class of structure, but such a schedule, even if its compilation were practicable, would be as cumbersome and as little helpful as an unabridged dictionary for the purpose of estimating first cost, because the items pertaining to the structure in hand at any time would be completely lost among the irrelevant items.

However, it is believed that all the items that would appear

[1] "Hand Book of Cost Data," by Halbert P. Gillette, second edition, 1910. The Myron C. Clark Publishing Co., Chicago and New York, 1854 pages, $5.

[2] A paper by Frank R. Ford, in Annals of the American Academy of Political and Social Science of January, 1911. Reprinted in Foster's "Engineering Valuation of Public Utilities and Factories," pp. 89–94.

in such a comprehensive schedule can be separated into a small number of fairly distinct classes, and that it is worth our while to list these classes and give some little consideration to each class. Accordingly we shall proceed to classify and to consider the classes one by one.

The expenditures which constitute first cost of structure—all the expenditures of investigation, promotion, and construction (or purchase and installation)—may be grouped under the following heads:

1. Labor (§41).
2. Real estate, materials, supplies, manufactured parts, and domestic animals (§42).
3. Transportation (§43).
4. Rights, rents, taxes, and insurance (§44).
5. Interest on capital tied up during investigation, promotion, and construction (or purchase and installation) (§45).
6. Contingencies (§46).
7. Outstanding data (§47).

Some consideration will now be given to the expenditures which are included in each of the foregoing groups.

41. Labor.—This term is here intended to include expenditures for labor of all kinds, mental as well as physical. Therefore for the present purpose the cost of labor will be considered to include the cost of the work of the expert and of the professional man as well as of the skilled and the unskilled laborer. The cost of labor of domestic animals which are hired or rented is also included here.

Labor is purchased by wages, hire, salaries, fees, commissions, brokerage, premiums, profits, percentage, or lump sum.

Wages are paid for skilled and unskilled labor including foremen, according to the number of hours on duty.

Hire or rent is paid for the labor of hired domestic animals, and will or will not vary with the amount of labor performed, according to the time unit in the hiring agreement.

Salaries are paid to officials, professional assistants, superintendents, draftsmen, bookkeepers, timekeepers, storekeepers, and clerks.

Fees and commissions are paid for such services of architects, doctors, engineers, lawyers, real-estate experts and other experts, as are not covered by salaries.

Commissions or brokerage or premiums may be paid for the work of obtaining the capital needful to meet first cost.

The profit, percentage, or lump sum, which comes to a contractor as a result of fulfilling the terms of a contract, includes the recompense for his labor.

Any condition which lengthens the period of creation of a structure increases the sum paid for salaries. In some cases the condition which protracts the creation period will increase not only the sum paid for salaries, but also the sum paid for wages and construction plant. For example, laying new rails on an operating track is comparatively expensive because of the time lost by men and plant, due to passing trains. Again, in erecting a trestle for a railroad not yet in operation, the time of men and forces may be devoted continuously to carrying the work forward; whereas if the trestle is being built for a track in operation, a considerable part of their time may be given over to temporary work for the safe passage of trains and to undoing the temporary work to make ready for continuing the work of construction proper.

42. Real Estate, Materials, Supplies, Manufactured Parts, and Domestic Animals.—*Real Estate.*—If the lands required for the site and operation of the structure are bought outright, the price paid is a part of the first cost of the structure. If existing improvements are bought with the land the price of the improvements is added to that of the land, whether or not the improvements can be used for the purpose in hand.

Materials.—This item includes all materials whatsoever of which the structure is composed, all materials which are used temporarily in the work of construction (for example, scaffolding), and all materials which are used in any necessary preliminary work (such as preparing the site of the structure).

Supplies.—Under this head are included those materials or things which are used up or consumed in furtherance of the creation of the structure; as, for example, water, oil, fuel, power, stationery and other office supplies, etc.

Manufactured Parts.—These comprise machines, tools, devices, articles of furniture, and the like, which are bought ready made and installed either for the purpose of forwarding the work of creating the structure or as component parts of the structure.

Domestic Animals.—Horses and mules are often bought outright for use in the work of creating a structure, and so sometimes are other animals.

Salvage Deducted.—In the case of real estate, materials, manufactured parts, and domestic animals employed in the work of creating the structure and not used up, the salvage value (§50) of such after use should be deducted from their cost and their net cost, only, entered as a part of first cost of structure. (See Chapter V.)

43. Transportation.—Expenditures may be incurred for the transportation of men, domestic animals, materials, supplies, manufactured parts, messages, etc. Cost of transportation is dependent firstly on the distance over which the thing is transported, and secondly on the character of available transportation facilities—whether water, rail, street, highway, mountain road or trail, or a combination of two or more of these. A short trail may equal a long railroad in the matter of exacting transportation charges.

One reason for the use of wood for original trestles and bridges and the later use of steel for renewals, on railroads traversing timbered regions, lies in the change in transportation facilities brought about by the opening of the railroad itself. Cost of transportation is not infrequently a deciding factor in the choice of structure for a proposed service, or in the choice of design or material, or of both, for a given structure.

44. Rights, Rents, Taxes, and Insurance.—*Rights.*—Rights include royalties on patented articles and processes, cost of charters, franchises, permits, licenses, rights of way over land, etc.

Rents.—Rents are paid for the use of grounds and existing structures such as offices, storage buildings, etc., and for leased construction plants, etc. Rental covers interest, taxes, insurance, depreciation, and profits, pertaining to the thing rented. The rental per day or per month is likely to be higher for a shorter than for a longer renting period.

Taxes.—Taxes, including assessments, may be levied on the structure or parts thereof before the beginning of operation. Yearly taxes vary with time and place, from less than 1/2 per cent. to more than 1–1/2 per cent. of the first cost of the structure. Precise data for computing taxes in the locality of the proposed structure may be obtained from the tax officers therein.

Insurance.—Premiums are paid for fire, casualty, fidelity, marine, title, employer's liability, and other insurance. In

some states the employer is subject to special laws relating to employer's liability.

It is reasonable that a contractor be insured against risk of loss. If the work of construction is executed under the most common form of contract, the contractor risks his own labor and a part of his capital as well, for the cash cost of executing the work under his contract may turn out to be equal to or even greater than the sum received under the contract. The cost of such insurance, which the contractor usually includes under his item of profits, is properly chargeable to first cost of structure.

Similarly, those who buy bonds which are issued to finance the construction of a structure, or those who contract to dispose of such bonds at a fixed price, run some risk of loss, and the issuers of the bonds pay insurance under the name of "bond discount." If such bonds are secured by a mortgage or other security which lacks nothing that the enterprise can reasonably give, and if further such bonds bear a rate of interest equal to that obtainable at the time of sale, such time being reasonably well chosen, on first-class commercial long-time securities, then the discount at such time of sale, if the sale is prudently made, is a proper charge for the insurance. (Good public policy requires that discount on bonds be treated in accounting as coordinate with, rather than subordinate to, first cost of structure, and accordingly that records of interest on, and amortization of, discount be kept coordinately with records of interest on and amortization of first cost. By this segregation foolish or fraudulent discounts are more likely to be prevented or recognized; and in any case all interested parties are continually reminded of the price being paid for a deficiency in financial standing. But the total yearly charge for interest and amortization will be the same whether we calculate that due to discount separately or otherwise, and since it simplifies somewhat the work in some of the following chapters to combine discount and first cost under the one head "first cost" we have included discount as an item under first cost.)

45. Interest on Capital Tied up During Investigation, Promotion, and Construction (or Purchase and Installation).—The formulas for cost of service, given in Table VII, account for the interest on capital only from the time beginning with first practicable operation of the structure. The interest, if any, which has accrued at the beginning of the first practicable operation of the structure, is properly a part of the first cost of the structure.

The interest on each payment on first cost of structure should be computed not for the interval between the date of payment and the earliest practicable date of operation, but for the interval between the date of first having the money on hand and the earliest practicable date of operation. Reasons for this rule are as follows:

The money to meet each payment on account of first cost must be on hand when the payment falls due. To have money on hand it must be withdrawn from the owner's other investments or borrowed. To withdraw or borrow a large sum of money requires time; and the owner must estimate the due date of each major payment and estimate the time it will take him to withdraw or borrow the money to meet the payment. Prudence requires that he adopt an early due date and allow a generous time for withdrawing or borrowing the money, for it is an act of common sense to provide against a conjunction of unfavorable circumstances; but since it is only in the exceptional case that all the circumstances turn out to be unfavorable, the exercise of prudence will, in the great majority of cases, result in having the money to meet the payment on hand a considerable time before it is paid out.

To make a separate withdrawal or borrowing to meet each minor payment is impracticable. Consequently to meet the minor payments expected to fall due within the next month or so, one withdrawal or borrowing is made and the money kept on hand to be paid out little by little.

Again, circumstances may virtually require that the entire capital for the cost of construction be on hand before the beginning of construction. If money is to be obtained from the sale of bonds, prudence may dictate that the bonds be offered for sale some time in advance of the need for the money, owing to a favorable condition of the bond market.

While the money is thus held on hand the owner is foregoing the interest on it if it has been obtained by withdrawal, or is paying interest for it if it has been borrowed or obtained from the sale of bonds; and in any case the interest is properly a part of the first cost of structure, provided that in case the money is obtained by withdrawal the interest rate should be no higher than that at which the money could be borrowed. For example, suppose that A withdraws $10,000 from other investments, paying him 10 per cent., to meet a payment on first cost of a struc-

ture, and that he could have borrowed the money readily at 8 per cent. The interest on this sum from the date of withdrawal to the first practicable date of operating the structure should be computed at 8 per cent., not 10 per cent., because the use of 10 per cent. money when 8 per cent. money is to be had is foolish, and the wasted 2 per cent. is rightly a loss to the owner and not a legitimate charge against the structure.

Sometimes the money held on hand can be placed in a safe depository which will pay a small rate of interest on such checking accounts; and when such is the case the interest received from the depository should be deducted from the interest foregone or paid by the owner for the money while on hand, and the remainder, only, entered as a part of the interest item of first cost. For example, for the purpose of making a betterment, a railroad sells one thousand \$1000 bonds, bearing 5 per cent. interest, at par. The betterment is immediately begun, and is completed in one year, and the money, lying in a bank paying 2–1/2 per cent. interest, is checked out from time to time during the year, a little at a time. If the average time during which all the money remains on deposit is one-half year, the net interest, $1,000,000 \times (0.05 - 0.025)1/2 = \$12,500$, is logically an item of the first cost of the betterment.

Interest on tied-up capital may be a matter of commanding importance. For example, a railroad may be constructed in one year, say, except for a single bridge which requires two years for completion by methods which give a minimum cost to the bridge itself. If all parts of the work are begun at the same time, the first cost of the other portions will be charged with the interest lost or paid during the extra year required for the building of the bridge. It may therefore be wise to hasten the construction of the bridge, at the expense of increasing its first cost, in order to save a part or all of the one year's interest on the money tied up in the remainder of the road. Similarly, to shorten the time of construction, a long tunnel may be attacked through shafts as well as at both ends, the resulting increase in the cost of the tunnel itself being much more than offset by the saving in interest on capital tied up in it and in the other parts of the road. Thus we see that shortening the building period of a structure may tend to decrease the interest element of first cost not only of that particular structure but, more important still,

also of a group of structures which with the particular structure constitute a structure of higher order.

It will occur to the reader that interest on tied-up capital can be reduced to a minimum by carrying on the work with utmost speed and so timing the beginning of work on each part of a structure as to bring all parts to completion simultaneously. For the construction of each component part of a complex structure, there are a beginning time, a method, and a speed of construction which together promise a minimum cost of construction exclusive of interest on tied-up capital. Beginning earlier, employing a quicker method, or working with greater speed will, it is true, decrease the item of interest on tied-up capital, but at the same time will increase some other item, or items, of cost by a sum which may be greater than the decrease in interest. For example, the most favorable time for erecting the steelwork of a bridge on a railroad under construction is, as far as the cost of the steelwork itself is concerned, after the track has been laid to the site, for then the material and the erecting plant can be brought to the site by rail. It is a common practice to lay track to each bridge in turn, and use the track for bringing up the steel and plant. This method, of course, delays track-laying, because the track-laying beyond each bridge must await the completion of the bridge; but the elimination of these delays would, in many cases, increase the item of transportation by a sum greater than the corresponding saving thereby effected in the item of interest on tied-up capital. It is for the engineer to study each case, and determine the time, method, and speed of construction, which, taken together, will give the minimum first cost including interest on tied-up capital. In connection with this paragraph, the reader is advised to read an article entitled "A Method of Determining Time of Performance of Work, with Special Application to Grading," written by G. L. Bennett and published in *Engineering and Contracting*, May 13, 1914, pp. 555–557, in which article is described a graphical method of determining the economic duration of construction of a proposed structure.

46. Contingencies.—Despite all care some items of cost are likely to be omitted from the estimate of cost of a very complex structure. Unforeseen changes in conditions, which affect the requirements of construction or the elements of cost thereof, may increase the required quantities or the unit prices. So it frequently happens that the actual cost of a structure is greater than

the sum of the estimated costs of items under the first five heads of the schedule of §40. To offset such possible deficiency in the estimate it is customary to add to the estimated cost some sum under the designation "contingencies."

Sometimes a contingent sum is added to one or more of the sub-totals of the estimate of first cost; sometimes a contingent sum is added only to the total estimate of first cost; and sometimes the two methods are combined.

The contingent sum in any case is computed by taking some chosen percentage of the total or sub-total to which it is to be added. The percentage in any given case may be as low as zero or as high as 20 per cent. or higher, according to the experience and judgment of the estimator, the character of the structure, and the conditions under which it is to be built. For example, a contractor who has in the past lost considerable money owing to deficiencies in his estimates will tend to set a relatively high percentage for contingencies, while the contractor who has lost little or nothing from such cause will tend to place a relatively low percentage for contingencies. Contingencies should be comparatively low for a simple structure which will be built in a short time under weather, market, and other conditions which promise to remain constant; and comparatively high in the case of a large, complex structure of which the building will take a long time for which the conditions promise to be uncertain.

47. Outstanding Data.—Under the preceding classes of items of first cost we have taken account only of those data to which are assigned definite numerical values. Now there may be some data which have a recognized bearing on the economic rating of the structure and yet cannot be satisfactorily appraised in dollars. For example, of two types of highway bridge proposed for a given site, let us say that No. 1 can be built in six months and No. 2 in 12 months, and assume that the traveling public will be materially inconvenienced throughout the period of construction. What is the present justifiable expenditure to save the traveling public from six months' material inconvenience? This question is in no case easy to answer. However, in few, if any, cases need the question be answered. Ordinarily the only question that need be answered is the following: Is it justifiable to pay $.... (a stated sum) to save the public from six months' material inconvenience? And this question is answered with comparative ease. It is always far easier to decide whether or

not you will take a thing at a named price than to decide on the maximum sum that you would give for it.

"Outstanding data" is the term applied in this book to all data which have a bearing on the economic rating of a structure but are not assigned any monetary values.

Contingencies and outstanding data are always to be distinguished by the fact that some definite numerical value is assigned to contingencies, while no numerical value is assigned to an outstanding datum.

In the following paragraphs are given examples of data which resist numerical evaluation and, if not evaluated, belong to the group called outstanding data.

Extra Cost of Operating During the Construction Period.—When a railroad bridge is being renewed, the cost of operating trains may be increased by the interruption to regular train movements and the extra precautions taken to prevent accident to trains in the vicinity of the bridge.

Loss of Net Income Due to Interruption to Operation.—If operation is interrupted during construction to the extent of decreasing the output of the plant of which the structure under construction is a part, there is a loss of net profit chargeable to the construction. When a washout or a wreck occurs on a railroad, passage of trains is stopped, and every hour of congestion may mean a net loss of thousands of dollars. In such a case the choice falls upon that temporary structure which, while not unsafe, can be completed in the shortest possible time, all other considerations to the contrary notwithstanding.

Accident to Operation, or to Property or Persons of Outside Parties, Due to Work of Construction.—This risk of expense should be considered, for in some kinds of construction work, as that involving the use of explosives, it becomes an important matter. Of course the design of the structure and the plan of work should provide against such accidents, and the cost of such provision should be estimated. But since such provision can seldom do more than lessen the probability of accident—cannot eliminate the possibility—there is in any case a residual risk which should be considered although incapable of being evaluated in dollars.

Accident to the Structure Itself or to the Construction Plant During the Period of Construction.—Such accident may require a duplication of part of the material or construction plant or a repetition of a part of the work, and protract the period of construction.

This risk is sometimes properly given great weight though not evaluated, as in the case of making economic comparison between two bridges proposed for crossing a river subject to high floods, of which one requires and the other does not require false works.

CHAPTER V

SALVAGE VALUE

This chapter defines "scrap value," "wearing value," "salvage value," and "depreciation;" discusses some conditions affecting salvage value; distinguishes cases in which salvage value is fixed by seller and buyer from cases in which it is fixed by a third party, and cases in which there is a second-hand market from cases in which there is no second-hand market; gives the derivation of four previously existing formulas for computing salvage value and introduces a new formula which takes account of earnings; applies all the formulas to some numerical examples and comments on the results and the formulas; and, finally, points out that any result found by formula must be weighed with all pertinent outstanding data.

A. Definitions

48. Scrap Value.—When a structure is no longer capable of performing service as a structure, or when the serviceability which is still possessed by a structure is not in demand, the structure has no value except the values of the materials and parts of which it is composed. The scrap value of a structure is the sum of the market prices of its component materials and parts, less the cost of their preparation for and transportation to market.

For example, when an old highway bridge is condemned to make way for its successor, the sounder timbers may be sold for some temporary use and the rods, bolts, and washers sold to be directly used or remelted. If the old bridge sells for $25 to a buyer who takes the bridge away at his own cost, the scrap value of the bridge is $25.

Scrap value may decrease or increase as the structure grows older (see §55).

49. Wearing Value.—Of the total sale value of a structure at a given time, that part which exists by virtue of the capability of the structure to render service as an entirety, is the "wearing value" of the structure at that time.

A structure may at the time of offering for sale be in condition to render service as a structure—may have remaining serviceability—yet, if no one can be found to pay anything for such ser-

viceability the wearing value at that time is zero. Thus when an iron bridge, which is no longer of use to the owner, is offered for sale, the wearing value of the bridge at that time is zero if no one can be found to buy the bridge for use as a bridge, even if the structure is practically new.

50. Salvage Value.—The "salvage value" of a structure is the price for which it is sold second-hand. The relation between salvage, wearing, and scrap values is indicated in the following equation:

$$\text{salvage value} = \text{wearing value} + \text{scrap value}. \qquad (46)$$

In most cases salvage value is fixed directly by buyer and seller; but there are some cases, usually important cases, in which salvage value is fixed by a third party. We shall discuss the two groups of cases separately.

51. Depreciation.—The term "depreciated value" in this book always means salvage value. The term "depreciation" in this book always has the meaning implied in the following equation:

$$\text{first cost} - \text{depreciation} = \text{salvage value}. \qquad (46a)$$

B. Salvage Value Fixed by Buyer and Seller

52. Price Fixed by Buyer and Seller.—Ordinarily in the sale of an article second-hand between private parties, the seller gets the biggest price that he thinks he can get, and the buyer pays the smallest price that he thinks the seller will accept.

When the seller has no competition and the buyer cannot well do without the article, the seller is likely to demand and obtain an abnormally large price. On the other hand, when the buyer has no competition and is under no constraint to buy, and the seller is pressed for money or has no further use for the article, the buyer is likely to obtain the article for a price abnormally low. Under such conditions the price mutually agreed upon will depend not only on whether the seller or buyer is in the tight place, but also on the tightness of the place, and this dependence is independent of other conditions named farther on.

If both buyer and seller have active competition neither buyer nor seller is at a disadvantage, and the price agreed upon may be called a "market price." Such market price is usually considered a fair price.

53. Conditions Affecting Salvage Value.—Salvage value is affected by the following conditions:

1. Reason for selling.
2. Market price of like structure new.
3. Condition of structure.

And if the sale of structure implies removal the following are to be added to the foregoing conditions:

4. Structure standard or special.
5. Structure massive or articulate.
6. Location of structure.

The bearings of these conditions on salvage value are considered in the following six sections.

54. Reason for Selling.—The price which can be obtained for a second-hand structure depends to a great extent on what the buyer thinks is the owner's reason for selling. It is reasonable to expect a higher price for a partly worn engine when it is part of a plant and business to be sold on account of the death or retirement of the owner than when it is offered for sale with the object of putting in its place a new, like engine; because in the latter case, and not in the former, suspicion is cast on the efficiency or durability of the engine offered. Suspicion is the factor chiefly responsible for the fact that the second-hand price of a little-used article in the open market is usually from 30 to 50 per cent. less than first cost.

55. Market Price New.—A structure which may be bought ready made in the market has a market price—in fact two market prices, a retail price and a wholesale price. When such a structure is bought and installed ready for use, but not yet used, its salvage value is probably a maximum. Such maximum salvage value cannot, except under conditions named below, be expected to exceed the wholesale price which obtains at the time of installation.

If the market price of similar new structures is greater (or less) at the time of second-hand sale than when the second-hand structure was bought new, the salvage value may be expected to be higher (or lower) than it would be if the market price new did not change.

The market price new may advance between the date of purchase new and the date of second-hand sale so far even as to cause the salvage value to exceed first cost. For example, steel rails bought at $16 per ton and used for years have brought a higher figure at second-hand sale because at the time of second-hand sale new rails had advanced to $28 per ton. And in the case of land

the salvage value is quite often much greater than the first cost.

Market price new is affected by fluctuations in prices of labor and materials, and also by the ratio of supply to demand. Demand may rise and fall with some regularity with the rounds of the seasons, as is the case with agricultural implements and machinery; with less regularity with waves of good and hard times; suddenly rise at the outset of a great constructional enterprise, and as suddenly fall at its completion; or change with changes in the tastes and mode of life of the people. Demand for any given type of structure may be cut off partially or entirely by the appearance of an improved type of structure designed for the same kind of service. A structure is said to be "obsolescent" when the demand for it is falling, owing to the introduction of improved structures, and "obsolete" when the demand has ceased entirely.

While in most cases it is out of the question to arrive by formal calculation at the future market price new of a structure, nevertheless it is necessary to take into account the possibilities of market changes, and weigh these along with other conditions when estimating salvage value.

The effect of change in market price new on salvage value is independent of other conditions which affect salvage value.

56. Condition of Structure.—Of two structures otherwise alike the one which is in the better condition—which possesses the greater number of units of serviceability—will have the greater salvage value. Thus salvage value may decrease through neglect of structure as well as through use. For example, the salvage value of a harvesting machine which is left unprotected in the field between harvests falls more between than during harvest seasons.

57. Structure Standard or Special.—(The following remarks are pertinent if the sale implies removal of the structure from its present site.) Of two structures otherwise alike, that one which is modified to meet peculiar local conditions will have a lower salvage value than the other which is standard (*i.e.*, designed for normal conditions). A half-worn, small, steel highway bridge, if square, may bring a salvage value much greater than scrap value, but if skew, scrap value only. The salvage value of a trestle which carries a railroad across a depression can seldom

be greater than scrap value, because the topographic conditions are rarely the same at any two sites.

58. Structure Massive or Articulate.—(This point has a bearing on salvage value only when the sale implies removal of structure to a new site.) If the cost of separating and putting together without injury the readily handled parts of a structure is great compared with the cost of the parts themselves, the structure is here said to be "massive." An "articulate" structure is one the parts of which can be separated and rejoined at a cost small compared with the cost of the parts themselves. According to these definitions, structures of earth, stone, concrete, and the like are massive structures, while machines are articulate in a high degree.

The maximum salvage value of a massive structure (where sale implies removal) may be little more than scrap value, except in the case of a structure which can be moved economically as a unit; while the salvage value of an articulate structure may approach, or even in some cases exceed, first cost. For example, the track on an embankment may sell second-hand for more than half its first cost, while the embankment cannot as a rule be sold at all; and a large structure of wood will have a greater salvage value when parts are joined by bolts and nuts than when spiked together.

59. Location of Structure.—Location may have a marked effect on salvage value if sale of structure implies removal. The salvage value of a more or less isolated structure is less than it would be if the structure were situated at its nearest market, by the cost of transportation to such market. Thus a nearly new engine which would bring a good price at an industrial center may, if located at the far end of a rough trail or long mountain road, as at a mine, find no taker when offered as a gift.

C. Salvage Value Fixed by Third Party

60. When Prices are Fixed by Third Party.—Not infrequently there arise cases in which the seller and buyer of a second-hand structure voluntarily agree to accept such price as may be fixed by a third party who is chosen by them to appraise the value of the structure. For example, one or more members of a firm who own an industrial plant may wish to retire from the firm, and it may be agreed that the continuing partners shall pay to the retiring partners a sum to be computed from the fractional inter-

est of the retiring partners and the value of the plant as determined by a disinterested appraiser who is acceptable to all parties.

Also there are cases in which the seller and buyer of a second-hand structure are required by law to accept such price for the structure as may be fixed by commission or court. For instance, when a railroad company requires for its use a strip of land, and the company and the owner of the strip cannot agree on a price to be paid by the company, the price may be fixed by appraisers chosen by the court. And so, also, when a municipality acquires by condemnation proceedings the plant of a water company which supplies the municipality, the price paid is that finally fixed by a court.

61. When there is a Second-hand Market.—When there is a second-hand market for the structure which is under appraisal, and the structure is to go to that market, the appraiser can ascertain the market price, estimate the cost of preparing and transporting the structure to market, subtract this estimated cost from the ascertained market price and so obtain the "fair price" of the structure. But when the structure is not to go to market, but remain where it is and continue its service, the market price for similar structures is not a fair price for the structure in question; for, if the buyer had to go to the market for a similar second-hand structure, he would have to pay, in addition to the offered market price, the profit of the market together with the cost of transportation from the market and installation. See further, under §62.

62. When there is no Second-hand Market.—Large industrial plants and public utility plants are so individual and so seldom sold in their entirety that there is no market for them and consequently they have no market price.

It may be suggested that there is a market for each of the parts composing any large plant, and that the sum of the market prices of the component parts must be the market price of the structure. But such suggestion ignores the fact that there may be a second-hand market for a structure under some conditions of sale and yet no second-hand market for the same structure under other conditions of sale. For example, let us consider the sale of a 4-inch, cast-iron water main which has served for 20 years as a part of an operating water-supply plant. First, consider that the pipe is to be detached from the plant and taken up

and sold. For this condition of sale there is, usually, a second-hand market for the pipe because second-hand cast-iron pipe is bought and sold generally. (There may be no salvage value under the conditions named, because the cost of taking up and marketing the pipe may be equal to or even greater than the market price.) Secondly, consider that the pipe is to be sold along with the plant and to be continued in operation with the plant. For this condition there is no second-hand market for the pipe, and consequently there can be no second-hand price for it; and of course not even a seeming market price for the whole plant can be obtained by the method suggested above.

So the appraiser who is selected to set a fair price to be paid, say, to a water company by a municipality for an operating water-supply system which is to be continued in operation, cannot execute his assigned task merely by consulting market reports. The reader naturally will wish to know by what method an appraiser *can* execute his assigned task in such cases. Several different methods have been used. To present and discuss such is the purpose of books on Valuation, titles and chapter headings of some of which are given in Appendix B. In the following sections of this chapter the discussion will be confined to those simpler methods which have been reduced to formulas.

63. Depreciation Formulas.—Any formula by which it is proposed to compute the salvage value or depreciation of a structure is usually called a "depreciation formula." In the following sections are presented, with their variations, five such formulas:

1. The "straight-line" depreciation formula (§64).
2. The "sinking-fund" depreciation formula (§65).
3. The "Matheson" depreciation formula (§66).
4. The "Gillette" depreciation formula (§67).
5. The "equal profit ratios" depreciation formula (§68).

It will appear in §67 that the straight-line formula and the sinking-fund formula are forms which the Gillette formula takes in special cases; but since the Gillette formula is the more recent it is presented after the other two. The last formula on the list above is the latest. The formulas are applied in three examples in §69, and briefly discussed in §70.

64. Straight-line Depreciation Formula.—This formula is based upon the assumption that the yearly depreciation is equal to the total drop in value during the life of the structure, divided by the life in years.

Let A = age, in years, for which salvage value is sought;
L = life of structure, in years;
C = first cost of structure (§39);
C_A = salvage value of structure at age A years (§50);
$\% C_A$ = per cent. ratio of C_A to C;
C_L = value of structure at end of life of L years;
f = yearly depreciation;
$\% f$ = per cent. ratio of f to C;
F_A = depreciation of structure up to age A years;
$\% F_A$ = per cent. ratio of F_A to C.

From the assumption and notation above we obtain the following formulas.

Yearly depreciation is

$$f = \frac{C - C_L}{L}. \tag{47}$$

Yearly depreciation, per cent., is

$$\% f = 100 \frac{C - C_L}{CL}. \tag{47a}$$

Depreciation up to age A years is

$$F_A = A \frac{C - C_L}{L}. \tag{47b}$$

Depreciation, per cent., up to age A years is

$$\% F_A = 100 \frac{A(C - C_L)}{CL}. \tag{47c}$$

Salvage value at age A years is

$$C_A = C - A \frac{C - C_L}{L}. \tag{47d}$$

Salvage value, per cent., at age A years is

$$\% C_A = 100 \left(1 - A \frac{C - C_L}{CL}\right). \tag{47e}$$

See §§69 and 70.

65. Sinking-fund Depreciation Formula.—This formula is based on the assumption that the depreciation up to any age, A, is equal to the amount, at the end of A years, of the sinking fund established (actually or in imagination) to offset the total life depreciation of the structure; in other words the formula is based on the assumption that depreciation and amount of amortization fund at any given time are equal.

Let A = age, in years, for which salvage value is sought;
C = first cost of structure;
C_A = salvage value of structure at age A years;
$\% \, C_A$ = per cent. ratio of C_A to C;
C_L = value of structure at the end of its life of L years;
F_A = depreciation of structure up to age A years;
$\% \, F_A$ = per cent. ratio of F_A to C;
d_L = sinking-fund deposit required to redeem \$1 at the end of L years;
z_A = amount of sinking fund in which \$1 is deposited yearly for A years.

The following formulas are based upon the assumption and notation given above.

Depreciation up to age A years is

$$F_A = (C - C_L)d_L z_A. \tag{48}$$

Depreciation, per cent., up to age A years is

$$\% \, F_A = 100 \frac{C - C_L}{C} d_L z_A. \tag{48a}$$

Salvage value at age A years is

$$C_A = C - (C - C_L)d_L z_A. \tag{48b}$$

Salvage value, per cent., at age A years is

$$\% \, C_A = 100\left[1 - \frac{(C - C_L)d_L z_A}{C}\right]. \tag{48c}$$

See §§69 and 70.

66. Matheson Depreciation Formula.—This formula differs from the straight-line formula of §64 in this, that, whereas with the latter the depreciation each year is taken as a constant percentage of first cost, with the former the depreciation each year is taken as a constant percentage of the salvage value at the beginning of that year. For example, if first cost is \$100 and the so-called rate of depreciation is taken as 5 per cent., then by the Matheson formula at age one year the salvage value is 100 − 5 = \$95; during the second year the drop in value is 5 per cent. of \$95, or \$4.75, and the salvage value at the end of the second year is 95.00 − 4.75 = \$90.25, and so on. Evidently this method gives a decreasing yearly depreciation; and the salvage value will become zero only at the end of time, if the structure have no scrap value.

Let A = age, in years, for which salvage value is sought;
C = first cost of structure;
C_A = salvage value at age A years;
$\% \ C_A$ = per cent. ratio of C_A to C;
C_L = value of structure at the end of life of L years;
F_A = depreciation of structure up to age A years;
$\% \ F_A$ = per cent. ratio of F_A to C;
k = ratio of depreciation in any one year to the salvage value at the beginning of that year. k is constant.

From the foregoing assumption and notation we construct the following tabulation:

Year	Yearly depreciation	Salvage value at end of the year, C_A		Total depreciation up to the end of the year
1st	kC	$C-kC$	$=C(1-k)$	kC
2d	$kC(1-k)$	$C(1-k)-kC(1-k)$	$=C(1-k)^2$	$C-C(1-k)^2$
3d	$kC(1-k)^2$	$C(1-k)^2-kC(1-k)^2$	$=C(1-k)^3$	$C-C(1-k)^3$
..	...	.. .	. .	...
Ath	$kC(1-k)^{A-1}$	$C(1-k)^{A-1}-kC(1-k)^{A-1}$	$=C(1-k)^A$	$C-C(1-k)^A$

From the tabulation we write the following formulas:

Depreciation during the Ath year $= kC(1-k)^{A-1}$. (49)

Depreciation up to the age A years is

$$F_A = C - C(1-k)^A. \tag{49a}$$

Depreciation, per cent., up to age A years is

$$\% \ F_A = 100\,[1-(1-k)^A]. \tag{49b}$$

Salvage value at age A years is

$$C_A = C(1-k)^A = C\left(\frac{C_L}{C}\right)^{A/L}. \quad \text{(See Eq. 49}e\text{)} \tag{49c}$$

Salvage value, per cent., at age A years is

$$\% \ C_A = 100\,(1-k)^A. \tag{49d}$$

Rate of depreciation, so called, is

$$k = 1 - \sqrt[A]{\frac{C_A}{C}} = 1 - \sqrt[L]{\frac{C_L}{C}}. \tag{49e}$$

See §§69 and 70.

67. Gillette Depreciation Formulas.—In his book, "Cost Data" (2d ed., 1910, p. 36), H. P. Gillette sets forth, as the basis for developing his formulas, the following principle:

"The owner of a second-hand machine [structure] is entitled to such a price for it as will enable the purchaser to go on with its use and produce each unit of product at as low a cost as the average unit cost of production would be during the entire life of the machine [structure]."

The same principle, less formally and somewhat incompletely expressed, is advanced by C. S. Burns on p. 80, Vol. LXIV, of the Transactions of the American Society of Civil Engineers, where he discusses the subject of waterworks valuation.

The development of the formula which follows, while based on the foregoing principle, differs from that given in "Cost Data" in so far as seemed needful to conform to the notation and form of presentation followed in the preceding pages of the present book.

Let A = age, in years, for which salvage value is sought;

C = first cost of structure;

C_A = salvage value of structure at age A years;

$\% \, C_A$ = per cent. ratio of C_A to C;

C_L = value of structure at the end of its life of L years;

d_L = yearly deposit in sinking fund required to redeem \$1 in L years;

d_{L-A} = yearly deposit in sinking fund required to redeem \$1 in $L - A$ years;

F_A = depreciation of structure up to the age A years;

$\% \, F_A$ = per cent. ratio of F_A to C;

M = yearly cost of maintenance (§83) for L years;

M'' = yearly cost of maintenance after first A years;

Q = yearly cost of operation (§81) for L years;

O'' = yearly cost of operation after the first A years;

r = ratio of yearly interest and other fixed charges to first cost, C (§78);

r'' = ratio of yearly interest and other fixed charges (for second-hand structure) to C_A;

U = yearly output of structure during life;

U'' = yearly output of structure after the first A years;

v_L = unit cost of service (§72) for life L years;

v_{L-A} = unit cost of service for the last $L - A$ years of life.

From Eqs. C3 (Table VII) and 67 we have

$$v_L = \frac{1}{U}[(C - C_L)d_L + Cr + O + M].$$

To him who buys the structure at the age of A years, the average unit cost of service during the $L - A$ remaining years of life will be

$$v_{L-A} = \frac{1}{U''}[(C_A - C_L)d_{L-A} + C_A\, r'' + O'' + M''].$$

According to the principle stated at the beginning of this section, v_{L-A} should equal v_L. Hence

$$\frac{1}{U''}[(C_A - C_L)d_{L-A} + C_A\, r'' + O'' + M''] = \frac{1}{U}[(C - C_L)d_L + Cr + O + M]. \qquad (50a)$$

Solving this equation for C_A we obtain the following:

Salvage value at age A years is

$$C_A = \frac{\frac{U''}{U}[(C - C_L)d_L + Cr + O + M] + C_L d_{L-A} - (O'' + M'')}{d_{L-A} + r''}. \qquad (50b)$$

Salvage value, per cent., at age A years is

$$\%\, C_A = 100\frac{C_A}{C}. \qquad (50c)$$

Depreciation up to age A years is

$$F_A = C - C_A. \qquad (50d)$$

Depreciation, per cent., up to age A years is

$$\%\, F_A = 100\frac{C - C_A}{C}. \qquad (50e)$$

See §§69 and 70.

Special Case I.—When the salvage value at the end of life is zero, and yearly output and yearly operating cost are uniform over the life of the structure, C_L = zero, $U'' = U$, and $O'' = O$; and introducing these values into Eq. 50b we obtain the following:

Salvage value at the age A years is

$$C_A = \frac{C(d_L + r) + M - M''}{d_{L-A} + r''}. \qquad (50f)$$

Salvage value, per cent., at age A years is

$$\%\, C_A = 100\frac{d_L + r + \frac{M - M''}{C}}{d_{L-A} + r''}. \qquad (50g)$$

Eq. 50g is virtually that to which Mr. Gillette has given the name "unit cost depreciation formula."

Special Case II.—When yearly output, yearly cost of maintenance, yearly cost of operation, and the ratio of interest and other fixed charges to first cost are uniform over the life of the structure, $U'' = U$, $O'' = O$, $M'' = M$, $r'' = r$. Placing these values in Eq. 50b we obtain

$$C_A = \frac{(C - C_L)d_L + Cr + C_L d_{L-A}}{d_{L-A} + r}. \tag{50h}$$

When, also, the "other fixed charges" are zero and the interest rate is the same for capital and sinking fund, Eq. 50h reduces to Eqs. 50i and 50j as follows. Substituting for d_L its value $\frac{s-1}{s^L - 1}$, for d_{L-A} its value $\frac{s-1}{s^{L-A} - 1}$ (where s = amount of \$1 for one year), and for r its value $s - 1$, we find that Eq. 50h reduces to

$$C_A = C - (C - C_L)\frac{s^A - 1}{s^L - 1}. \tag{50i}$$

Since $\frac{s^A - 1}{s^L - 1} = \frac{s-1}{s^L - 1} \cdot \frac{s^A - 1}{s - 1} = d_L z_A$, we obtain from Eq. 50i the following:

Salvage value at age A years is

$$C_A = C - (C - C_L)d_L z_A. \tag{50j}$$

This equation, as the reader will see, is the "sinking-fund" depreciation formula (Eq. 48b).

Special Case III.—If to the provisions of special case II we add that interest is to be ignored, Eq. 50h becomes

$$C_A = \frac{(C - C_L)\frac{1}{L} + C_L \frac{1}{L-A}}{\frac{1}{L-A}}. \tag{50k}$$

Reducing Eq. 50k, we obtain the following:

Salvage value at age A years is

$$C_A = C - \frac{A(C - C_L)}{L}. \tag{50m}$$

It should be observed that Eq. 50m is the "straight-line" depreciation formula (Eq. 47d).

68. Equal Profit Ratios Formula.—Sometimes the demand for the service of a structure is expected to increase from the begin-

ning of operation, because (1) of the spreading recognition of the convenience of the service, and the people's readjustment of their mode of life to take advantage of it; and (2) because of the growth of the population, as for example in the case of a new waterworks for a small town, or a railroad in an undeveloped territory. Now, for some reasons it is not economical to build a structure to meet simply the service demand of the first year, and for other reasons it is not economical to build to meet the demand expected 50 or 60 years hence. Though the problem of determining for what future time it is economical to build now, comes under the head of engineering economics and the engineer may be called upon to solve it, the solution of the problem is here passed by as being beyond the scope of this book. However, whatever may be the precise number of years for which a large and complex structure is built, one thing is certain, namely, during the early years of operation the potential capacity of the structure may not be fully employed, and we may say a part of the invested capital may thus be necessarily idle for a time. And it may be that the yearly earnings from the sale of service during the earlier years will be less than the yearly cost of service. During each of the early years when the earnings do not meet all the cost of service, not only is there no money to pay a profit, but the owner must borrow money (from himself or others) to pay that portion of cost of service which the earnings do not cover. Thus he must wait for the profit belonging to the earlier years and pay interest on the money borrowed, until such time as the excess earnings become great enough to reimburse him.

Many structures are planned and built with the foreknowledge that the yearly earnings of the early years will not pay the corresponding yearly costs of service, on the reasonable expectation that the earnings of later years will exceed the yearly costs by sums which will pay not only a reasonable yearly profit for the later years, but, also, all deferred profits and the money borrowed during the earlier years together with interest thereon.

How is a fair price to be determined for a structure from which the earnings have not been sufficient to pay yearly costs of service and the yearly profits up to the time of the proposed sale? Evidently a price computed by any formula which ignores earnings will be greatly to the disadvantage of the seller, who, unless earnings are taken into account, may be constrained to whistle for his profits while he pays out of his own pocket the debts

incurred for money to help to keep the structure operating. It is believed that no formula can give a fair second-hand price unless it recognize earnings; and in this belief a new formula (Eq. 50r) is offered, based on the following principle:

A fair price for a second-hand structure is such as will cause the ratio of equivalent uniform yearly profit to investment to be the same for the seller as for the buyer.

Let L = the life of structure, in years;
A = age, in years, for which salvage value is sought;
z_A = amount of an A-year sinking fund in which \$1 is the end-of-year deposit.

For the seller:

Let b' = equivalent uniform yearly profit for the A years (§81);
e' = equivalent uniform yearly earnings for the A years (§81);
y_A = equivalent uniform yearly cost of service for the A years (§107);
C = first cost of structure (to the seller);
C_A = salvage value of structure at age A (this is the first cost to the buyer);
d_A = end-of-year deposit in the sinking fund required to redeem \$1 at the end of A years;
r' = ratio of yearly interest and other fixed charges (for the seller) to the seller's first cost, C (§78);
O' = equivalent uniform yearly cost of operation for the A years (§81);
M' = equivalent uniform yearly cost of maintenance for the A years (§83).

For the buyer:

Let b'' = equivalent uniform yearly profit for the $L - A$ years (§81);
e'' = equivalent uniform yearly earnings for the $L - A$ years (§81);
y_{L-A} = equivalent uniform yearly cost of service for the $L - A$ years (§107);
C_A = first cost of structure (to the buyer);
C_L = salvage value of structure at age L years;

d_{L-A} = end-of-year deposit in the sinking fund required to redeem \$1 at the end of $L - A$ years;

r'' = ratio of yearly interest and other fixed charges (for the buyer) to the buyer's first cost, C_A (§78);

O'' = equivalent uniform yearly cost of operation for the $L - A$ years (§81);

M'' = equivalent uniform yearly cost of maintenance for the $L - A$ years (§83).

For the seller, whose investment is C, the ratio of equivalent uniform yearly profit to investment is b'/C; and for the buyer, whose investment is C_A, the ratio of equivalent uniform yearly profit to investment is b''/C_A. According to the principle stated above,

$$b'/C = b''/C_A. \tag{50n}$$

From the notation above,

$$b' = e' - y_A,$$

and

$$b'' = e'' - y_{L-A};$$

and from Eq. A2 of Table VII (introducing the primes here for present purposes),

$$y_A = (C - C_A)d_A + Cr' + O' + M',$$

and, similarly,

$$y_{L-A} = (C_A - C_L)\, d_{L-A} + C_A r'' + O'' + M''.$$

Substituting these values of b', b'', y_A, and y_{L-A} in Eq. 50n, we have

$$\frac{e' - [(C - C_A)d_A + Cr' + O' + M']}{C} = \frac{e'' - [(C_A - C_L)d_{L-A} + C_A r'' + O'' + M'']}{C_A}, \tag{50o}$$

from which we obtain

$$C^2_A + C_A z_A[e' - (O' + M') - C(d_A - d_{L-A} + r' - r'')] = Cz_A[e'' - (O'' + M'') + C_L d_{L-A}]. \tag{50p}$$

Let $2g$ = the coefficient of C_A, and m = the right-hand member of the equation; then

$$C^2_A + 2gC_A = m. \tag{50q}$$

Completing the square

$$C^2_A + 2gC_A + g^2 = m + g^2.$$

yearly output for the whole life. Substituting for y_{L-A} and y_L their values in terms of amortization, interest and other fixed charges, operation, and maintenance (see the schedule of §72 and line A of Table VII), we obtain,

$$\frac{1}{U''}[(C_A - C_L)d_{L-A} + Cr'' + O'' + M''] = \frac{1}{U}[(C - C_L)d_L + Cr + O + M]. \quad (50z)$$

Eq. 50z is the same as Eq. 50a. Thus we find that the equal profit ratios formula reduces to the Gillette formula, for the special case in which the profits, other than those paid in the form of interest on first cost, are zero and the earnings are proportional to the output.

69. Examples of Use of Formulas.—The following numerical examples are solved by each of the five formulas.

Example 1.—We have the following data pertaining to a structure. Required the salvage value at the age of five years.

First cost	$10,000
Life	10 years
Salvage value at end of life	0
Interest rate on capital and amortization fund	5 per cent.
Yearly cost of operation and maintenance:	
During the first five years	$1,000
During the last five years	$2,000
Yearly output (units of service):	
During the first five years	1,000
During the last five years	2,000
Yearly earnings from sale of service:	
During the first five years	$3,000
During the last five years	$6,000

Solution by Straight-line Formula.—This formula (Eq. 47d) is

$$C_A = C - A\frac{C - C_L}{L}.$$

For this example $C = 10{,}000$, $A = 5$, $C_L = 0$, and $L = 10$; hence the salvage value is

$$C_A = 10{,}000 - 5\,\frac{10{,}000 - 0}{10} = \$5000. \quad \textit{Answer.}$$

The reader will observe that this formula ignores interest, output, and cost of operation and maintenance, as well as earnings. See under Examples 2 and 3 below, and §70.

Solution by Sinking-fund Formula.—This formula (Eq. 48b) is

$$C_A = C - (C - C_L)d_L z_A.$$

For this example

$$C = 10{,}000,\ C_L = 0,\ A = 5,\ L = 10;$$

$d_L = d_{10} = 0.0795$ (opposite "10," under "5 per cent.," Table E), and $z_A = z_5 = 5.526$ (opposite "5," under "5 per cent.," Table C).

Hence
$$\begin{aligned} C_A &= 10{,}000 - (10{,}000 - 0)(0.0795)(5.526) \\ &= \$5606.83. \quad \textit{Answer.} \end{aligned}$$

It will be observed that this formula ignores output, interest on capital, and cost of operation and maintenance, as well as earnings. See under Examples 2 and 3 below, and §70.

Solution by Matheson Formula.—This formula (Eq. 49c) is

$$C_A = C\left(\frac{C_L}{C}\right)^{A/L}.$$

In this example $C = 10{,}000$, $L = 10$, $A = 5$. According to the data, C_L is zero, but if we substitute zero for C_L in the formula we shall come out with $C_A =$ zero. The basis of the Matheson formula implies that salvage value can be zero only at an infinite age. Therefore to defer to the letter of the implication and at the same time practically to keep to the spirit of the data, let us say that $C_L = \$0.01$.

Hence
$$\begin{aligned} C_A &= 10{,}000\left(\frac{0.01}{10{,}000}\right)^{5/10} = 10{,}000(0.000001)^{1/2} \\ &= 10{,}000(0.001) \\ &= \$10. \quad \textit{Answer.} \end{aligned}$$

It will be noticed that the Matheson formula does not take into account output, interest, cost of operation and maintenance, or earnings, and to this extent is in the same class as the straight-line formula.

Solution by Gillette Formula.—This formula (Eq. 50b) is

$$C_A = \frac{(U''/U)[(C - C_L)d^L + Cr + O + M] + C_L d_{L-A} - (O'' + M'')}{d_{L-A} + r''}.$$

In this example

$$\begin{aligned} U'' &= 2000, \\ U &= (1000 \times 5 + 2000 \times 5)/10 = 1500, \\ L &= 10,\ A = 5,\ C = 10{,}000,\ C_L = \text{zero}, \end{aligned}$$

$d_L = d_{10}$ $= 0.0795$ (opposite "10," under "5 per cent.," Table E),

r $= r'' = 0.05,$

$O + M$ $= (1000 \times 5 + 2000 \times 5)/10 = 1500,$

d_{L-A} $= d_5 = 0.181$ (opposite "5," under "5 per cent.," Table E),

$O'' + M'' = 2000.$

Hence

$$C_A = \frac{\frac{2000}{1500}[(10000-0)(.0795) + 10000(.05) + 1500] + 0(.181) - 2000}{0.181 + 0.05}$$

$$= \frac{1.333(795 + 500 + 1500) - 2000}{0.231}$$

$= \$7474.74.$ *Answer.*

The reader will notice that the Gillette formula takes account of all elements of cost of service, but does not recognize earnings. See also under Examples 2 and 3 below, and §70.

Solution by Equal Profit Ratios Formula.—This formula (Eq. 50r) is

$$C_A = \sqrt{m + g^2} - g$$

where (Eq. 50s)

$$g = (1/2)\,z_A[e' - (O' + M') - C(d_A - d_{L-A} + r' - r'')],$$

and (Eq. 50t)

$$m = Cz_A[e'' - (O'' + M'') + C_L d_{L-A}].$$

In this example

$L = 10, A = 5,$

$z_A = z_5 = 5.526$ (opposite "5," under "5 per cent.," Table C),

e' $= 3000,$

$O' + M'$ $= 1000,$

C $= 10{,}000,$

d_A $= d_{L-A} = d_5 = 0.181$ (opposite "5," under "5 per cent.," Table E),

r' $= r'' = 0.05,$

e'' $= 6000,$

$O'' + M'' = 2000,$

C_L $=$ zero.

Therefore

$$g = (1/2)(5.526)[3000 - 1000 - 10{,}000(0.181 - 0.1810. + 05 - 0.05)]$$
$$= 5526;$$

$$
\begin{aligned}
g^2 &= & 30{,}536{,}676 \\
m &= 10{,}000(5.526)[6000 - 2000 + 0(0.181)] = & 221{,}040{,}000 \\
m + g^2 &= & 251{,}576{,}676
\end{aligned}
$$

$$\sqrt{m + g^2} = 15{,}861.17,$$
$$g = 5{,}526.00,$$

and $$C_A = \$10{,}335.17. \quad \textit{Answer.}$$

It is observable that the equal profit ratios formula takes account of the earnings as well as of all the elements of cost of service. The output enters only indirectly—through the earnings. See under Examples 2 and 3 below, and §70.

It may be interesting to see what will be the ratio of yearly profit to investment for the seller and for the buyer for the salvage value computed by each of the five formulas above. For the seller the yearly cost of service is (Eq. A2 of Table VII)

$$y_A = (C - C_A)d_A + Cr' + O' + M'$$

and the ratio of yearly profit, b', to investment, C, will be (Eq. 50n)

$$b'/C = (e' - y_A)/C$$

where e' = seller's yearly earnings.

For the buyer, the yearly cost of service will be

$$y_{L-A} = (C_A - C_L)d_{L-A} + C_A r'' + O'' + M'',$$

and the ratio of his yearly profit, b'', to his investment, C_A, will be

$$b''/C_A = (e'' - y_{L-A})/C_A,$$

where e'' = buyer's yearly earnings.

When the salvage value is computed by the straight-line formula

$$
\begin{aligned}
y_A &= (10{,}000 - 5000)(0.181) + 10{,}000(0.05) + 1000 \\
&= 2405,
\end{aligned}
$$

and the profit ratio for the seller is

$$
\begin{aligned}
b'/C = (e' - y_A)/C &\doteq (3000 - 2405)/10{,}000 \\
&= 5.95 \text{ per cent.}
\end{aligned}
$$

Likewise

$$y_{L-A} = (5000 - 0)(0.181) + 5000(0.05) + 2000$$
$$= 3155,$$

and the profit ratio for the buyer is

$$b''/C_A = (e'' - y_{L-A})/C_A = (6000 - 3155)/5000$$
$$= 56.9 \text{ per cent.}$$

In the same way the profit ratios have been computed for each of the salvage values found by means of the four other formulas above. All the salvage values and corresponding profit ratios are tabulated below.

YEARLY PROFIT RATIOS RESULTING FROM SALVAGE VALUES COMPUTED BY FIVE DIFFERENT FORMULAS FOR EXAMPLE 1

Formula	Salvage value at age five years	Ratio of yearly profit to investment, for the	
		Seller, per cent.	Buyer, per cent.
Straight-line	$5,000.00	5.95	56 9
Sinking-fund	5,606.83	7.05	48 2
Matheson	10.00	−3.08	39,976.9
Gillette	7,474.74	10 4	30.4
Equal profit ratios	10,335.17	15.61	15.61

It must be borne in mind that the profit ratios given in the table above are for the particular case stated at the beginning of this section; and that were the data of the example changed throughout, the ratios would be changed.

Example 2.—The data of this example are as follows:

First cost	$75
Life	10 years
Salvage value at end of life	$5
Interest on capital	5 per cent.
Interest on amortization fund	4 per cent.
Yearly cost of operation and maintenance	$10
Yearly output (units of service)	300
Yearly earnings (from sale of service)	$30

By means of each of the five formulas used in Example 1 above, the salvage value of the present example was computed for each age, one year, two years, and so on, up to 10 years. (The computations, being similar to those of Example 1 above, are omitted). The results for each formula were then plotted (Fig. 7), using years

for abscissas and the corresponding salvage values as ordinates. The plotted points were then connected by lines as shown. Thus from the Matheson curve, for example, the salvage value of the structure for which data are given above, at age five years, is about 26 per cent. of the first cost.

It should be remembered that the curves of Fig. 7 result from the data of this particular example. Were we to change the

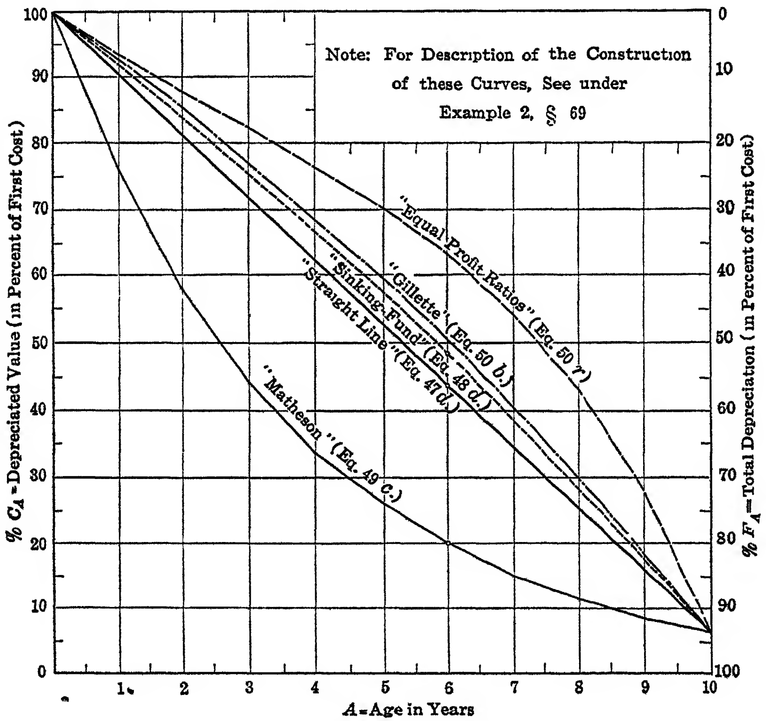

Fig. 7.—Depreciation formulas compared by application to a particular case.

salvage value at end of life from $5 to zero, or any other value, all five curves would be changed. If, however, we should change only the earnings, only one of the curves—that for equal profit ratios—would be changed.

For this example the following profit ratios have been computed on the basis of salvage value at age six years, by the method illustrated in Example 1.

YEARLY PROFIT RATIOS RESULTING FROM SALVAGE VALUES COMPUTED BY FIVE DIFFERENT FORMULAS FOR EXAMPLE 2

Formula	Salvage value at age six years	Ratio of yearly profit to investment, for the	
		Seller, per cent	Buyer, per cent.
Straight-line....	$33 00	13 2	35.7
Sinking-fund...	36 32	13.9	29.8
Matheson...	14.80	9 6	114.5
Gillette..	37.69	14 2	27.8
Equal profit ratios	47 42	16 1	16.1

Example 3.—This example is the same as Example 2, above, except that this has not uniform output and earnings.

First cost	$75
Life	10 years
Salvage value at end of life	$5
Interest on capital	5 per cent.
Interest on amortization funds	4 per cent.
Yearly cost of operation and maintenance	$10
Yearly output (units of service):	
During the first six years	100
During the last four years	500
Yearly earnings (from sale of service):	
During the first six years	$10
During the last four years	$50

For the data of this example the salvage values computed by the five formulas used in Example 1, together with the corresponding profit ratios, are given in the following tabulation:

YEARLY PROFIT RATIOS RESULTING FROM SALVAGE VALUES COMPUTED BY FIVE DIFFERENT FORMULAS FOR EXAMPLE 3

Formula	Salvage value at age six years	Ratio of yearly profits to investment, for the	
		Seller, per cent.	Buyer, per cent
Straight-line.....	$ 33.00	−13.5	96.2
Sinking-fund	36.32	−12 8	84.8
Matheson	14.80	−17.1	250.0
Gillette	101.00	0.2	12.2
Equal profit ratios	123.59	4.8	4 8

It will be noticed that the salvage values above for the first three formulas are the same as for the same formulas in the preceding example.

70. Comments on Depreciation Formulas.—For reasons given in §68, the equal profit ratios formula is believed to be based

directly on the correct principle underlying all logical derivations of salvage value, and is expressed in general terms. As is shown in §68, this formula reduces to the Gillette formula if two assumptions are made. Furthermore, as was shown in §67, the straight-line and sinking-fund formulas are two forms to which the Gillette formula reduces for certain special conditions. It follows that these four formulas, *equal profit ratios*, *Gillette*, *sinking-fund*, and *straight-line*, have the same rational basis, and differ from one another only in the number of special assumptions which must be made. The equal profit ratios formula is the most general, and the straight-line the least. If the principle underlying the equal profit ratios formula be accepted as fair, logically the salvage values computed by this formula must be accepted as fair; and the salvage values computed by the Gillette, sinking-fund, and straight-line formulas must be accepted as fair when, and only when, the special conditions described in §§67 and 68 can be assumed to hold.

The *Matheson* formula is based on the idea that yearly depreciation grows less and less as the age of the structure increases. This is the law of second-hand market price, which usually takes the greatest drop during the first year of life of structure, and drops less and less, year by year, thereafter. Hence this formula may be useful in predicting second-hand market price. But, as remarked in §54, second-hand market price is greatly depressed, especially for structures nearly new, by the factor of suspicion. Since this factor is non-existent in cases where salvage value is determined by a disinterested party, the use of the Matheson formula is not justified for such determination.

The new formula (§68), called *equal profit ratios* formula, is thus shown to furnish the most general solution of the problem of salvage value. It is important, however, that we bear in mind the fact that *no formula can take into account all the conditions and circumstances which may have legitimate influence on the salvage value of a structure;* and that, accordingly, the salvage value computed for any case by this formula must be used with judgment, that is, must be considered in connection with all such pertinent factors as are not represented in the formula. In other words, the computed salvage value must be considered coordinately with all outstanding data (§47). The formula assumes that all financial and engineering plans for the creation and operation of the structure have been made and carried out with reason-

able intelligence and absolute honesty and good faith. It is the duty of the appraiser to eliminate from the data the effects of any fraud or incompetence on the part of the promoters, constructors, or operators, before applying the formula.

The reader is referred, for further discussion of salvage value, to the books listed in Appendix B, and of these he is advised to read first Hayes's "Public Utilities."

CHAPTER VI

ELEMENTS OF YEARLY COST OF SERVICE

This chapter defines "original structure," "renewals," and "replacements" and gives a schedule of elements of "yearly cost of service;" discusses these elements under the main heads of amortization, interest and other fixed charges, operation, maintenance, and outstanding data; and presents working formulas all of which are collected in Table VII in Part V.

A. Definitions and a Schedule

71. Original Structure, Renewals, and Replacements.—If for estimating purposes we imagine a series of like structures, one taking the place of another as that other wears out, we call the first of the series the "original structure," the second of the series we call the "first renewal," the third we call the "second renewal," and so on; and this is so whether or not the first structure of the imagined series is considered as following an actual worn-out structure or series of actual worn-out structures. Thus in the following pages, original structure is the term applied to the first of a particular series of structures held in mind at one time for estimating purposes. "Original life" is the life of such original structure; "original cost" and "original salvage value" are respectively the cost and salvage value of such original structure. "Renewal life," "renewal cost," and "renewal salvage value" are respectively the life, cost, and salvage value of a renewal. In the following pages the several lives of a structure are assumed to be of equal length.

When an old or worn structure is superseded by a new structure of the same type, material, and capacity, the old structure is said to be "renewed," and the new structure is called a "renewal," as stated above. When a structure of some age, whether worn out or not, is superseded by a new structure which differs from the old in type, material, or capacity, the old structure is said to be "replaced," and the new structure is called a "replacement."

72. Schedule of Elements of Yearly Cost of Service.—The items of yearly cost of service may be grouped under heads and subheads of the following schedule. In this schedule, under "fixed charges" are grouped costs which are the same, or very nearly the same, whether the structure is operated or idle; while under

"running expenses" are placed those costs which exist only with the operation of the structure, not when the structure is idle. The separation of the elements of yearly cost of service into two grand divisions, one of which consists of fixed charges and the other those charges which arise and fluctuate with operation, is of obvious advantage in studying some phases of the economics of structures; otherwise the precise grouping of the elements and the names applied to the groups are of very little moment. In

A Schedule of Elements of Yearly Cost of Service Performed by a Structure

I. Fixed Charges	(Those yearly outlays which are necessary to maintain the structure idle but in good condition.)
I. Amortization (§73)	(A function of first cost (§§39 and 40), salvage value (§50), and interest rate on sinking funds (§75).
II. Interest on first cost, and other fixed charges.	
1. Interest on first cost of structure (§76).	
2. Other fixed charges (§77).	
Labor Materials, supplies, etc. Transportation Rights, rents, etc.	(That minimum required to maintain the structure idle but in good condition.)
II. Running Expenses.......	(All and only those expenses which come into existence with operation.)
III. Operation (§79).	
1. Labor.	
2. Materials, supplies, etc.	
3. Transportation.	
4. Rights, rents, taxes, and insurance.	
5. Interest on working capital (§80).	
6. Contingencies.	
IV. Maintenance (§82).	
1. Labor.	
2. Materials, supplies, etc.	
3. Transportation.	
4. Rights, rents, taxes, and insurance.	
5. Interest on working capital.	
6. Contingencies.	
V. Outstanding data pertaining to running expenses (§84).	

the problems considered in this book the important thing is to make sure that *all* the items of yearly cost are taken into the reckoning. For example, so far as the use of the schedule for the following problems is concerned, "amortization" and "interest on first cost" might as well be combined under one head, "capital

cost," and the items under "other fixed charges" and "maintenance" be included with the items under "operation," for which grouping there is precedent. Railroad expenditures are usually grouped in two grand divisions under the respective titles "fixed charges" and "operating expenses," the former of which includes interest on capital, and rents and taxes. (The complete schedule of operating expenses of railroads prescribed by the Interstate Commerce Commission has been given in various books, among which may be mentioned Raymond's "Elements of Railroad Engineering," Webb's "Railroad Construction," and Webb's "Economics of Railroad Construction."[1] These three books are named here for the reason that in each of them is presented a method of estimating the effect, on profits, of making any proposed change in the place, manner, or means of producing service, which method should be familiar to all engineers.)

The elements of the foregoing schedule are considered in the following sections.

B. Amortization

73. Amortization Described.—The meaning of "amortization" may best be presented to the reader who has given the subject no consideration through the aid of an example. Let us assume that

we have no capital;
we borrow $5000 at 5 per cent.;
we pay $5000 for a sawmill which will last 10 years;
the sawmill will be worth nothing at the end of 10 years;
we devote the mill to sawing for other persons;
out of the mill earnings each year we pay all running expenses, and the interest (0.05 × 5000 = $250) on the debt, and devote the remainder to personal expenses.

What is our financial situation at the end of the 10 years when the affairs of the mill are wound up? Our assets are then zero: all the earnings have been expended (none invested) and the mill itself is worth nothing. Our liabilities are $5000, for nothing has been paid on the principal of the debt. Hence the net result of the sawing business is that we owe $5000 and have nothing to pay the debt with. (If the mill at the end of 10 years were worth $100, say, instead of nothing, the net result would be: we owe $5000 and have $100 to pay on the debt, leaving a net debt of $4900.)

[1] These three books are published by John Wiley & Sons, New York.

It is plain enough that something is wrong with the financial planning exhibited in the example above. It is evident that we should have made provision for paying the $5000 debt at the end of the life of the mill. As the earnings of the mill are the only income (assuming the worn-out mill to be worthless), the earnings must provide the means of paying the $5000. We can fully provide for the payment of the $5000 by building up a sinking fund, thus: after paying the running expenses and the interest on the $5000 each year, further take from the year's earnings $5000d_{10}$ (d_{10} being the yearly deposit in the sinking fund required to redeem $1 at the end of 10 years) and deposit the same in a sinking fund. In Table E, under "5 per cent." and opposite "10," we find d_{10} to be equal to $0.0795. Therefore the annual deposit of 5000 (0.0795) = $397.50 will at the end of 10 years give us the amount, $5000, with which to wipe out the $5000 debt. (If the salvage value of the mill at the end of life were $100 instead of nothing, the annual deposit in the sinking fund would be (5000 − 100) (0.0795) = $389.55 instead of $397.50.)

The sinking fund in the example above is an "amortization fund," the annual deposit is the "yearly amortization charge," and this method of providing for the payment of the debt is called "amortization."

The principle is the same whether we are indebted for the original capital to some third person or to our own pockets. If we invest our own money in the mill, out of the mill earnings each year should be taken the sum required for deposit in the amortization fund, in order that at the end of the 10 years we may have back the undiminished capital, $5000.

74. Formulas for Yearly Cost of Amortization.—There are three cases to be considered, depending upon the relation between the period of service and the life of the structure.

Case I. When Stated Period of Service is Less Than One Life.—If the stated period of service be A years, and A be less than L, the life of the structure, the formula is

$$\begin{array}{c}\text{yearly cost of amortization}\\ \text{up to the age } A \text{ years}\end{array} = (C - C_A)\mathrm{d}_A \qquad (51)$$

where C = first cost of structure;

C_A = salvage value of structure at age A years;

d_A = deposit in sinking fund required to redeem $1 at the end of A years.

Fig. 8 shows graphically the decrease in d_A with increase of both time and interest rate.

Approximate Formula.—When the rate of interest on the amortization fund is zero, Eq. 51 reduces to

$$\begin{matrix}\text{yearly cost of amortization} \\ \text{up to the age } A \text{ years}\end{matrix} = (C - C_A)\frac{1}{A}. \qquad (51a)$$

This equation gives only approximate results when the interest rate is other than zero. Sometimes to save time in computation, formula 51a is used even when the interest rate is not zero, with the idea that the result will be sufficiently near the truth for the

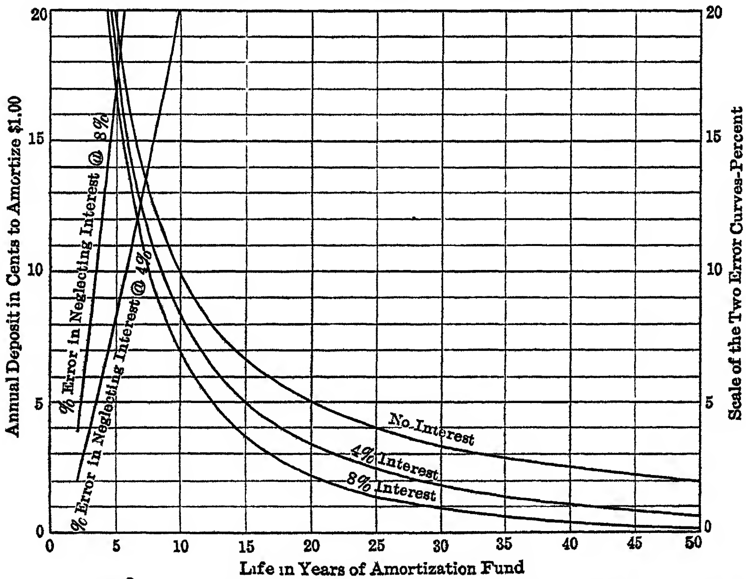

FIG. 8.—Effect of life and interest rate on yearly amortization charge.

purpose in hand. By an inspection of Fig. 8 we find that the yearly cost of amortization, as computed by this approximate formula, is as much as 50 per cent. too high if A = 11 years and interest is at 8 per cent., or if A = 20 years and interest is at 4 per cent. For A = five years and interest at 4 per cent., the yearly amortization cost is about 8 per cent. too large when computed by Eq. 51a. The reader will be able to find from Fig. 8, for given years and interest rate, the error resulting from the use of Eq. 51a instead of Eq. 51, and so can decide whether it is permissible to use the approximate formula under the given conditions;

and in fact he may, if he desires to do so, use Eq. 51a and correct the approximate result by subtracting therefrom the error indicated by Fig. 8 for the given years and interest rate. For 50 years and 8 per cent. the error due to using Eq. 51a instead of Eq. 51 is about 1100 per cent.

Example.—A machine costing $2000 is proposed for a five-year service. The salvage value of the machine at the age of five years is estimated at $1200. What is the yearly amortization charge for the service of the machine if sinking-fund deposits draw 6 per cent. compounded annually? $C = 2000$, $C_A = C_5 = 1200$. $d_A = d_5$, and is found in Table E (opposite "5" and under "6 per cent.") to be 0.1774 (or, in Fig. 8, to be 0.175 (check)). The yearly amortization is therefore

$$(C - C_A)d_A = (2000 - 1200)(0.1774) = \$141.92. \textit{ Answer.}$$

(By the approximate formula, Eq. 51a, the result would be

$$(C - C_A)\frac{1}{A} = (2000 - 1200)/5 = \$160. \quad \textit{Approximate answer.}$$

This result is too large by $18.08, or about 12–1/2 per cent. By inspection of Fig. 8 we find that with five years and 6 per cent. this error is about 13 per cent., thus checking the 12–1/2 per cent. computed directly from the two answers just above. Whether or not the approximate formula gives a satisfactory result in this example depends upon the uncertainty in the estimated quantities substituted in the formula. In this example an error of $100 in the estimated salvage value would introduce an error of about 12 per cent. into the answer computed by the correct formula, Eq. 51.)

Case II. When the Stated Period of Service is Equal to One or More Whole Lives.—If the stated period of service is a multiple of the life of the structure—if the period is L, or $2L$, or $3L$, and so on—the formula is

$$\begin{matrix}\text{yearly cost of amortization for}\\ \text{any whole number of lives}\end{matrix} = (C - C_L)d_L \qquad (52)$$

where L = life of structure;

C = first cost of structure (and of each renewal if there be any renewals);

C_L = salvage value of original structure (and of each renewal if there be any renewals);

d_L = annual deposit in sinking fund required to redeem $1 at the end of L years.

For a structure of everlasting life, $(C - C_L)d_L = 0$, because for this condition d_L is zero.

Approximate Formula.—The remarks on the approximate formula of Case I are directly applicable here if we replace the letter "A" therein by the letter "L."

Example.—A machine costing \$2000, and having an estimated life of 15 years, is proposed for a 15-year service. The salvage value of the machine at the end of its life is estimated to be \$100. What is the annual amortization charge if sinking-fund deposits draw 6 per cent. compounded annually? Substituting numerical values in Eq. 52 ($d_L = d_{15}$ is found in Table E to be 0.0430), we have

$$(2000 - 100)(0.0430) = \$81.70. \quad \textit{Answer.}$$

(By the approximate formula, $(C - C_L)/L$ (corresponding to Eq. 51a), the answer would be $(2000 - 100)/15 = \$126.67$, which is about 56 per cent. greater than the true answer, \$81.70. An error of \$100 in the estimated scrap value would in this example introduce into the answer an error of only about 5 per cent.)

Case III. When the Stated Period of Service is Equal to One or More Lives Plus a Fractional Life.—If the stated period of service is $n' (= n + A)$ years, where n is a multiple of L (one life of structure) and A is age (less than L), the yearly cost of amortization is not uniform throughout the period. For, during the first n years, the yearly cost of amortization is $(C - C_L)d_L$ (see Case II above), while during the last A years of the period it is $(C - C_A)d_A$ (see Case I above). If the actual yearly cost of amortization be not uniform over the whole n' years, we substitute for the actual series of yearly amortization costs an equivalent uniform series. The equivalent uniform yearly cost of amortization is equal to the annuity for n' years produced by a principal which is equal to the sum of the present worths (at the initial date of the service period) of the series of yearly amortization costs, $(C - C_L)d_L$, covering the first n years, and the other series of yearly amortization costs, $(C - C_A)d_A$, covering the last A years of the n'-year period. The present worth of $(C - C_L)d_L$ per year for n years is $(C - C_L)d_Lw_n$; and the present worth of $(C - C_A)d_A$ per year for A years, deferred n years, is $(C - C_A)d_A(w_{n'} - w_n)$, where w_n and $w_{n'}$ are respectively the present worths of \$1 per year for n years, and \$1 per year for n' years. Therefore the equivalent uniform

yearly cost of amortization
for n' ($= n + A$) years

$$= [(C - C_L)d_L w_n + (C - C_A)d_A(w_{n'} - w_n)]a_{n'}, \quad (53)$$

where C = first cost of original structure (and of each renewal if there be any renewals);
L = life of original structure (and of each renewal if there be any renewals);
A = age, in years, of structure existing at the end of the n'-year period;
C_A = salvage value at the age A years;
C_L = salvage value at the end of life;
d_A = annual deposit in sinking fund to redeem \$1 at the end of A years;
d_L = annual deposit in sinking fund required to redeem \$1 at the end of L years;
w_n = present worth of \$1 per year for n years;
$w_{n'}$ = present worth of \$1 per year for n' years;
$a_{n'}$ = annuity produced for n' years by a principal of \$1.

(Eq. 53 reduces to Eq. 51 when the stated period of service is A years, and to Eq. 52 when the period of service is L years or any multiple of L years.)

Example.—A machine costing \$2000, and having an estimated life of 15 years, is proposed for a 35-year service. (The 35-year service will require the full life of the original machine plus the full life of the first renewal plus five years of the life of the second renewal.) The estimated salvage value of the machine at age five years is \$1200, and at the age of 15 years, is \$100. What will be the (equivalent uniform) yearly amortization charge if interest on sinking-fund deposits is 6 per cent. compounded annually? From Eq. 53 the yearly amortization charge is

$$[(C - C_{15})d_{15}w_{30} + (C - C_5)d_5(w_{35} - w_{30})]a_{35}.$$

C = 2000,
C_{15} = 100,
C_5 = 1200,
d_{15} = 0.0430 (taken from Table E, opposite "15" and under "6 per cent."),
d_5 = 0.1774 (taken from Table E, opposite "5" and under "6 per cent."),
$w_{30} \doteq$ 13.67 (taken from Table D, opposite "30" and under "6 per cent."),

w_{35} = 14.50 (taken from Table D, opposite "35" and under "6 per cent."),
a_{35} = 0.0690 (taken from Table F, opposite "35" and under "6 per cent.").

Thus the yearly amortization charge is

$$[(2000 - 100)(0.043)(13.67) + (2000 - 1200)(0.1774)(14.50 - 13.67)](0\ 069) = \$84.73.$$

Answer.

(A shorter, though approximate, computation is as follows: taking the yearly cost of amortization as $(C - C_L)d_L$ (= \$81.70) for the first 30 years, and as $(C - C_A)d_A$ (= \$141.92) for the last five years, the *average* yearly amortization cost is

$$[(30 \times 81.70) + (5 \times 141.92)]/35 = \$90.30.$$

This approximate answer is about 6 per cent. greater than the true answer, \$84.73. The computation can be still further shortened by ignoring interest altogether, thus:

$$[(2000 - 100) + (2000 - 100) + (2000 - 1200)]/35 = \$131.43$$

which result is greater than the true answer, \$84.73, by about 55 per cent. (See under "approximate formula," Case I.)

75. Interest Rate on Amortization Funds.—The rate of interest on amortization-fund deposits will depend on the disposition made of the deposits. The annual deposit may be (1) placed in a savings bank or other similar institution which pays a moderate rate of interest; or (2) invested in productive business, either in the business in hand or in some other, and thus probably but not certainly earn a relatively high rate of interest; or (3) handed directly to the party from whom the capital has been borrowed, as an advance payment on the principal, which disposition is equivalent to placing the deposit in an absolutely safe repository to draw compound interest at the rate paid for the capital invested in first cost. In some cases the disposition of future deposits may be fixed by law or agreement. In all other cases it is reasonable, for the purpose of estimating the yearly cost of service of a future structure, to make the rate on deposits the same as the rate on capital invested in first cost of structure.

The effect of interest rate for amortization fund on the size of the yearly deposit required, is shown graphically in Fig. 8 (p. 77). For instance, the per cent. ratio of the yearly deposit to the amount of a 10-year amortization fund is 6.9 per cent. if fund

draws 8 per cent.; is 8.4 per cent. if fund draws 4 per cent.; and is 10 per cent. if fund draws no interest whatever.

C. Interest and Other Fixed Charges

76. Interest on First Cost of Structure.—When money is borrowed to build a structure, interest is paid year after year on the sum borrowed, until the borrowed money is paid back. The money may be borrowed on a note; in which case the interest paid is at the rate written in the note. The money may be obtained from the issue and sale of bonds. Bonds, like promissory notes, require that interest be paid at the rate written in the instrument. The rate written may be as low as 3 per cent. or lower, or as high as 8 per cent. or higher.

On a $1000 bond calling for 6 per cent. interest to be paid annually, the party issuing the bond must pay $60 a year to the purchaser of the bond or to his order, during the life of the bond.

Bonds are sold for what they will bring, and this may be less or more than the face value. But, whether the bonds are sold at a discount or at a premium, the interest to be paid each year to the bond holder is determined solely by the par value of the bond and the rate of interest stated therein.

If the first cost of structure is paid out of the pocket of the owner, interest must still be allowed on first cost, since it is to be supposed that this capital would earn interest elsewhere for the owner if not invested in the structure.

77. Other Fixed Charges.—To maintain a structure idle, but in good condition, may require expenditure for some or all of the following:

1. Labor (§41).
2. Materials, supplies, etc. (§42).
3. Transportation (§43).
4. Rights, rents, taxes, and insurance (§44).

For a very short period of idleness due to interrupted operation, all members of the operating force will remain under pay. For a longer period, a partial force will be maintained. And even for a maximum period of idleness, whether due to interrupted or deferred operation, a minimum of management, superintendence, and of professional, technical, skilled, and unskilled labor will be required to maintain the structure in good condition. (See §41.)

Materials, supplies, etc., will be required for those attentions,

repairs, and renewals which are made necessary by the ravages of time; and for the office and other uses of the maintenance force. (See §42.)

Transportation must be taken into account when considering the cost of maintaining an idle structure, though this item will seldom be large and often negligible. (See §43.)

For an idle structure there may be some expenditures for rights, and rents; and the items of taxes and insurance may be nearly or quite as large in some cases for the idle as for the operating structure.

It has been stated previously that it makes no difference in the computed yearly cost of service whether an outlay is put under the head of "other fixed charges," or under "maintenance," or under "operation." Therefore in any case of economic comparison of two or more structures, in which it requires time and effort to segregate the items, which, according to §72, belong to "other fixed charges" of a proposed structure, it is economical to omit the segregation and include such items under "operation" or "maintenance," or some under one and some under the other. It will be observed that in some of the examples of Chapter X the "other fixed charges" have not been segregated. However, it should not be forgotten that, in studying the economics of operating a given structure, it is necessary to segregate all fixed charges.

78. Formula for Yearly Cost of Interest and Other Fixed Charges.—The ratio of interest and other fixed charges to first cost is represented by r, so that with first cost denoted by C,

$$\begin{array}{c}\text{yearly cost of interest and}\\ \text{other fixed charges}\end{array} = Cr. \tag{54}$$

Example.—The first cost of a structure is \$10,000, on which interest is paid at 5 per cent. To maintain this structure idle, will require yearly for labor, materials, etc., \$200, and for taxes and insurance 1 per cent. on first cost. What is the yearly cost of interest and other fixed charges?

Interest rate (ratio)	0.05
Taxes and insurance (ratio)	0.01
Labor, etc., (ratio, 200/10,000)	0.02
r	0.08

$$Cr = 10{,}000 \times 0.08 = \$800. \quad \textit{Answer.}$$

Otherwise computed:

Yearly interest, taxes, and insurance,	
10,000 (0.05+0 01)	$600
Yearly cost of labor, etc.	200
Cr.	$800. *Answer.*

(It is evident that as far as total yearly cost of service is concerned the $200 for labor, etc., might as well be included under "cost of maintenance" and omitted from "other fixed charges.")

D. Operation

79. Operation.—In the schedule of §72 the expenditures attributed to operation (exclusive of maintenance) are given as follows:

1. Labor (§41).
2. Materials, supplies, etc. (§42).
3. Transportation (§43).
4. Rights, rents, taxes, and insurance (§44).
5. Interest on working capital (§80).
6. Contingencies (§46).

Items 1, 2, 3, 4, and 6 have been sufficiently discussed in the sections cited above. It should be said here, however, that the cost of raw materials for articles to be produced by the operation of a structure is no part of the cost of operation as the term is used in this book.

80. Interest on Working Capital.—In the formulas for cost of service which will be considered in Chapter VIII, "yearly cost of operation" is treated as a sum paid, or payable, at the *end* of the year. But payments are made for labor, materials, etc., etc., on account of operation, from time to time *during* the year. Hence "yearly cost of operation" in the following pages should include not only every such payment, but also the interest accrued on the payment at the end of the year. In computing this interest, we should recognize the fact that the money to make each payment during the year must be on hand some time before the date of payment (see §45).

81. Formulas for Yearly Cost of Operation.—If the yearly cost of operation of a structure is constant,

$$\text{yearly cost of operation} = O. \tag{55}$$

When the yearly cost of operation is variable the equivalent uniform yearly cost of operation is

$$O = a_L[O_1p + O_2p_2 + O_3p_3 + \ . \ . \ . + O_Lp_L] \tag{56}$$

where L = the number of years under consideration;
a_L = the annuity for L years produced by the principal \$1 (Table F);
$O_1, O_2, \ldots, O_L$ = the yearly cost of operation for the first, second, . . ., and the Lth years respectively;
$p, p_2, \ldots, p_L$ = present worths of \$1 due respectively 1, 2,. . , and L years hence (Table B).

Sometimes the computer is justified, by the saving of time effected thereby, in using instead of equivalent uniform yearly cost the *average* yearly cost which is

$$O = (O_1 + O_2 + O_3 + \ldots + O_L)/L \tag{57}$$

where the notation is the same as above.

Example.—The cost of operating a proposed pump is estimated to be as follows:

first year,	\$1740,
second year,	1740,
third year,	1740,
fourth year,	6400,
fifth year,	6400.

What is the equivalent uniform yearly cost if money is worth 6 per cent. compounded annually? The equivalent uniform yearly cost is computed by Eq. 56 which becomes for this case

$$O = a_5(O_1p + O_2p_2 + O_3p_3 + O_4p_4 + O_5p_5).$$

a_5 = 0.2374 (taken from Table F, opposite "5" and under "6 per cent."),
p = 0.9434 (taken from Table B, opposite "1" and under "6 per cent."),
p_2 = 0.8900 (taken from Table B, opposite "2" and under "6 per cent."),

and so on. Thus

$$\begin{aligned} O_1p &= 1740 \times 0.9434 = 1640\pm. \\ O_2p_2 &= 1740 \times 0.8900 = 1550. \\ O_3p_3 &= 1740 \times 0.8396 = 1460. \\ O_4p_4 &= 6400 \times 0.7921 = 5070. \\ O_5p_5 &= 6400 \times 0.7473 = \underline{4780.} \end{aligned}$$

Sum of products 14500.

Therefore $O = 0.2374 \times 14{,}500 = \$3440 \pm$. *Answer.*

(We have made the computation, above, according to the letter of the formula, but we might have taken advantage of the fact that in this example the first three yearly costs may be considered as deposits in a three-year sinking fund, and the last two yearly costs may be considered as the deposits in a two-year sinking fund deferred three years. The present worth of the former is 1740 w_3 and of the latter 6400 $(w_5 - w_3)$, where w_5 and w_3 are the present worths of \$1 a year for five years and \$1 a year for three years, respectively. Therefore $O = a_5[1740w_3 + 6400(w_5 - w_3)]$.

$w_5 = 4.212$ (opposite "5" and under "6 per cent." in Table D),

$w_3 = 2.673$ (opposite "3" and under "6 per cent." in Table D).

$1.539 = w_5 - w_3$.

Therefore

$$O = 0.2374(1740 \times 2.673 + 6400 \times 1.539)$$
$$= \$3440\pm. \quad \textit{Answer.}$$

The *average* yearly cost of operation (Eq. 57) in the example above, is $(3 \times 1700 + 2 \times 6400)/5 = \$3404$. This result, which ignores interest, is smaller than the true result, \$3440, by \$36 or about one per cent. Whether or not the use of Eq. 57 instead of Eq. 56 is justified in a given case depends on the degree of uncertainty in the estimated yearly costs of operation and the use to which the result is to be put.)

E. Maintenance

82. Maintenance.—This division of yearly cost, according to the schedule of §72, includes all outlays for repairs and upkeep which are directly due to operating the structure. The outlays for maintenance which would be incurred for the structure idle are covered by the term "other fixed charges" (§77).

In the schedule of yearly cost of service, §72, the outlays for maintenance are grouped as follows:

1. Labor (§41).
2. Materials, supplies, etc. (§42).
3. Transportation (§43).
4. Rights, rents, etc. (§44).
5. Interest on working capital (§80).
6. Contingencies (§46).

By way of supplementing the matter presented in the sections

cited above, it is proper to say that in some cases repairs are of special importance, as where parts can be had only from a distance, with the attendant delays which may interrupt operation and thereby reduce the output and raise the cost per unit of output. For instance, relative convenience of obtaining repair parts often determines which one of two or more makes of machine shall be purchased; and again, the relative convenience of obtaining job repairing is sometimes the deciding factor in making choice of location for a manufacturing business.

83. Formulas for Yearly Cost of Maintenance.—If the yearly cost of maintenance is constant, we have, simply,

$$\text{yearly cost of maintenance} = M. \tag{58}$$

If the yearly cost of maintenance is not uniform, the equivalent uniform yearly cost of maintenance can be found by Eq. 56 after replacing each "O" therein by an "M."

The remarks on the use of the *average* yearly cost of operation, in §81, can be made to apply here by replacing the word "operation" therein by the word "maintenance," and substituting for the letter "O" in Eq. 57 the letter "M."

The example of §81 will serve for this section also, if "operation" and "O" be changed to "maintenance" and "M," respectively.

F. Outstanding Data

84. Outstanding Data.—In estimating running expenses (operation and maintenance), as in estimating first cost, there may be some economic factors which, while readily recognized, cannot be appraised directly in dollars. Of course such factors cannot be included in the estimated running expenses, but they must nevertheless be enumerated and attached to and made a part of the estimate. As representative of such factors, the following will serve.

1. Risk of accident after construction, which may increase the cost of maintenance or of operation, or decrease the output.

2. Risk of damage to outside property or persons through operating the structure.

3. The outlook for the requisite supply of labor and materials for operation and maintenance (see last paragraph of §82).

Any economic factor which by choice or necessity the computer does not evaluate in dollars is in this book called an "outstanding datum." (See §47.)

CHAPTER VII

ESTIMATING

This chapter presents by example five methods of estimating cost; and discusses the preparation and use of estimating data, the effect of the laws of error on estimates, and the estimation of quantities of some particular types. References to published cost data will be found in Appendix D.

It is plain that all the elements of cost of service to be produced by a proposed structure, must be estimated in order to arrive at an estimate of future yearly, capitalized, or unit, cost of service (§§107–110). Thus it is in order to give some attention here to the principles and practice of estimating.

A. On the Practice of Estimating

85. Methods of Making Preliminary Estimates of Cost.—There are several methods of making a preliminary estimate of cost of a structure. Each branch of engineering may have one or more methods which are peculiar to itself, but some methods are applicable to all branches. Some of the methods of general application are illustrated in the following sections.

It will be apparent that selection of method should be based upon the character of the data, the use to which the results are to be put, the time or money available for the work of making the estimate, and the experience of the estimator.

The data given in the following methods have been inserted only for illustrative purposes and are not intended to be used of themselves as information. See §103 for methods of ascertaining future unit prices; and Appendix D for a list of books and periodicals which give cost data.

86. Method of Total Cost.—When an experienced builder inspects the plans and specifications for a proposed dwelling with a view to making a rapid estimate of its cost, he mentally passes in review all the dwellings that he has built, and notes that the dwelling which he built for Henry Hillhouse at a cost of $5500 is on the whole very like the proposed dwelling as to quantity and quality of materials and labor required; recognizes the fact that

prices are a little higher now than when the Hillhouse dwelling was built, and accordingly estimates the cost of the proposed dwelling at \$6000. Or, perhaps, he classifies the proposed dwelling as lying (as to cost) somewhere between two dwellings already built by him—one for Brown at a cost of \$5500, the other for White at a cost of \$7200; judges that the proposed dwelling will cost nearer \$7200 than \$5500, and estimates its cost at \$6500.

The method here exemplified is used more than all other methods of estimating. It is as quick if not quicker than any other method. The estimator applies the method with equal facility, if not with the same sureness, to the structure of which he has only a dim mental picture and to that of which he holds complete working drawings. This method is, or should be, always used as a check on cost estimated by other methods; and is universally employed by executives to test the work of subordinates for blunders. Furthermore, this method is more reliable in some cases than the more elaborate methods—in cases where the plans and specifications and conditions which have controlled the cost of several actual structures will control the cost of the proposed structure.

The data required by the estimator in this method are the total costs and general descriptions of type, dimensions, location, and market and construction conditions, of actual structures similar to the proposed structure. It is perhaps needless to add that the best results by the use of this method are to be expected from an estimator who has been intimately connected with the construction of the structures from which his data have been taken.

87. Method of Total Unit Cost.—The total unit cost is obtained by dividing the total cost of a structure by the number of units in the structure. The best unit for estimating purposes is that which gives the most nearly uniform total unit cost. For example, a builder calculates for each dwelling of each type built by him, the total cost per room, per opening, per square foot of floor area, per cubic foot of enclosed space, and per M of lumber used. That unit which gives the most nearly uniform total unit cost is the best unit to use for estimating purposes, if we disregard the time taken to make the estimate (evidently it will require more time to take off the bill of lumber than to count the rooms or compute the floor area).

Having chosen the cubic foot of enclosed space, let us assume,

as the most desirable unit for estimating purposes, the builder prepares a chart with an x-axis representing total cubic feet of enclosed space, and a y-axis representing total cost per cubic foot of enclosed space. On the chart he plots for each of the dwellings of one type built, a point of which the abscissa represents the total cubic feet of enclosed space, and the ordinate the total cost per cubic foot of enclosed space.

By joining the plotted points with straight lines, or by drawing a smooth averaging curve through the points, if this be deemed the better plan, he obtains a curve from which, for a given space enclosed in a proposed dwelling of the same type, he can obtain the probable cost per cubic foot to be enclosed. Multiplying the cost per cubic foot by the total number of cubic feet gives an estimate of the cost of the proposed dwelling.

The chart can be made in a few minutes on squared paper, and an estimate be made by its use in still less time.

For the application of this method of estimating, the estimator requires the data named in §86, together with the data for making the charts above mentioned.

88. Method of Parts.—By this method the structure is mentally divided into a greater or smaller number of parts; an estimate is made of the cost of each part; and the estimate of the cost of the structure obtained by summing.

For example, a builder, on looking over the plans and specifications for a proposed dwelling, recognizes that the dwelling is, in point of cost, in many parts like the corresponding parts of dwellings which he has completed, and in one or more parts quite different:

Parts same as in completed structures.	Excavation . .		$......
	Masonry.		$. ....
	Lumber...		$.
	Carpenter work. . .		$ . .
	Plastering.... . .		$... .
	Shingling.		$. . .
	Painting. . .		$......
			$......
Parts not the same as in completed structures.	Millwork.	$.. . .	
	Hardware.	$. . . .	
	Plumbing...	$.. ..	......
Total estimate of cost..............			$......

He divides the structure into two groups of parts as shown, and

estimates the cost of each part separately. For the parts which are the same as the corresponding parts in some structure completed by him, the estimating is quickly done by either of the preceding methods, from memory or from recorded data. For the other parts: he judges that the millwork will be 50 per cent. higher than usual on account of the better quality; he finds from a hardware merchant the relative cost of better-grade hardware to be about double the cost of the ordinary grade; and figures on a half more plumbing than in the completed house. Thus the estimate is made.

It is to be observed that this method separates the usual from the unusual, permitting special and undivided attention to be given to each unusual part.

This method calls for data from past experience on costs of parts similar to data required in the two preceding methods on costs of whole structures.

89. Method of Ratio of Whole to Part.—A builder analyzed his cost records pertaining to several completed frame dwellings and found that the per cent. ratios of items to total ran about as follows:

General item	Per cent. of whole cost
Masonry	5
Lumber	23
Carpenter labor	21
Plastering	9
Millwork	17
Plumbing	8
Shingling	6
Hardware	4
Painting	7
Total cost of dwelling	100

The builder estimated the lumber for a proposed frame dwelling at $960; his past experience, as represented by the schedule above, showed that the cost of lumber was about 23 per cent. of the total cost; consequently he computed the probable cost of the proposed dwelling at (100/23)960 which, in round numbers, is $4200.

The foregoing is to be taken merely as an example illustrating a method of estimating; the percentages given are not intended for information.

Of course the degree of confidence to be placed in an estimate made by this method will depend, in part, upon the number of

structures upon which the schedule is based, and the variations which the schedules of individual completed structures show in comparison with the adopted average schedule. In general it is to be expected that the larger the percentage of the item chosen as a basis, the more reliable will be the result; for instance, the lumber item (23 per cent.) in the schedule above should be more reliable as a basis than the plumbing item (8 per cent.). The number of items into which the whole cost should be divided for the purpose of compiling a percentage schedule will depend upon circumstances and the character of the original data.

It is to be supposed that the mason would incline to use the masonry cost, the plumber the plumbing cost, the millman the millwork cost, and so on, as a basis for computing the total cost of a dwelling from the schedule.

90. Method of Complete Analysis and Unit Costs.—This method carries the method of parts to a finer point. It calls for the separation not only of part from part, of the structure, but for each part separates raw material from labor, classifies the raw materials, and classifies the operations by which labor and plant convert the raw materials into the finished product, estimates the units of raw material in each class, and the units of labor required for each class of operation, and the cost for each unit of material and of labor (in other words, the unit costs).

For a complex structure this method takes more time than any other. It can be applied to a whole structure, but need be applied only to such parts of a structure as cannot be satisfactorily estimated by one of the shorter methods.

Example 1 (*Operations*).—A builder reads the plans and specifications for a proposed dwelling, and (let us say, in order to be brief) recognizes the fact that this proposed dwelling is in all ways, except one, the fellow of a dwelling already completed by him. The one exception is that the proposed dwelling site is such as to require the excavation of an unusually large quantity of earth which must be moved to an unusually great distance. He estimates the cost of all but the earthwork by one of the preceding methods. As the earthwork is beyond his experience, he cannot estimate its cost in lump satisfactorily, nor can he estimate the cost per cubic yard from total-cost-per-cubic-yard taken from his past experience because he has never hauled excavated material a long distance. On looking at the matter a little more closely he recognizes that some of the required earth-handling

operations are the same as in his experience. He thereupon lists the operations:

1. Loosening the earth.
2. Loading the earth.
3. Hauling the earth.
4. Dumping the earth.

He has had experience in loosening earth similar to the earth under consideration and knows the cost per cubic yard for loosening. From past experience he also knows the cost per cubic yard for loading, and the cost per cubic yard for dumping. He finds the cost per cubic yard for hauling by estimating the capacity of a wagon and the number of loads that a team can haul over the given distance in one day and divides the product of these into the cost of one team for one day. Adding together costs per yard for loosening, loading, hauling, and dumping, he obtains the total cost per cubic yard; and this multiplied by the number of cubic yards in the proposed excavation gives an estimate of the total cost of the earthwork. Adding this result to the estimated costs of the other parts of the work he obtains an estimate of the total cost of the proposed dwelling.

Example 2 (*Materials*).—A builder is asked to bid on the construction of a wooden grandstand the plans and specifications for which are furnished to him. He has had no experience with buildings of this character, and has no data regarding the costs of completed grandstands. So he reads the specifications, and then goes over the plans and "takes off the quantities" of materials of different kinds and classes. For the lumber, for instance, he fills out a list similar to the following:

Part	No of pieces	End dimensions	Length	Feet B M	Wastage	Total feet B M.	Price per M	Total cost
Mud sills	68	4 × 6	2′ 6″					
Longitudinal sills ..								
Cross sills								
Posts.								
Etc.								
Etc.								

In this list are entered braces, plates, purlins, rafters, siding, sheathing, seats, and so on, in addition to the parts which we have noted in the form above.

With the same care, bolts, spikes, nails, and other hardware

are estimated. The same process is applied to any other parts of the structure. The builder then estimates the man-days of labor required for each group of building operations, including those of transporting and unloading the materials, applies the unit prices for labor, and thus obtains an estimate of total cost of labor, and so on. Finally, by adding together the partial estimates, he forms an estimate of the total cost (to him) of the entire proposed structure. He adds to this, perhaps, some percentage to cover any small items of expense which may have been overlooked, and adding a percentage for "profits" he obtains his bidding price.

This method of estimating calls for data on the units of work in each operation, and the price per unit of work; and data on the conditions of past work which have had a bearing on the number of units and the price per unit.

91. Circumstances of Past Work Are Indispensable Data.—In the examples of methods of estimating costs of proposed structures, given in the preceding sections, it is assumed that the estimator drew upon his own experience for his estimating data; and it is there further assumed that the estimator carried in memory the circumstances which accompanied the building operations from which he obtained his data; and consequently the circumstances were not described, and the importance of the knowledge of the circumstances to the estimator was not made clearly apparent. It often happens that the estimator must rely upon recorded numerical data, derived either from his own experience or from that of others, or from both; and in such case it is necessary that he appreciate fully the unreliability of a record which gives numerical data of cost without descriptions of the qualifying circumstances.

If we know the number of cubic yards in a given excavation and the total cost of making the excavation, we can calculate the average cost per cubic yard. This computed unit cost appears at first glance to be a perfectly definite and usable quantity for one who is about to estimate the cost of a proposed excavation of which he has estimated the volume; for what is plainer than that estimated volume multiplied by cost per cubic yard taken from actual experience gives an estimate of total cost of the proposed excavation? It does, indeed, give *an* estimate, but one that is entirely untrustworthy, because cost of excavation does not depend solely upon the number of cubic yards excavated: it depends also upon the

nature and condition of the material excavated; the kind and condition of the excavating plant used; the intelligence and vigor with which the work is carried on; the shape of the excavation; the length of the haul; wages; and other conditions and circumstances. For example, rock costs more than earth, mainly because it requires a greater amount of labor or its equivalent to loosen it; wet earth requires more effort than dry, because it sticks to the shovels and vehicles, and in some cases, as in tunnels and foundation pits, the cost of excavating soft material greatly exceeds that of solid rock because of the water that has to be pumped to keep the excavation in workable condition; in an excavation of considerable extent machine work is usually cheaper than hand work; very shallow excavations, deep trenches, shafts, wells, and tunnels, all cost more per cubic yard than excavations in similar material, which, due to more favorable shape, present fewer hindrances to the free movement of the forces; work is usually done less rapidly when laborers are scarce than when they are plentiful; work done under contract is as a rule pushed more vigorously than work done by days' labor; higher wages and higher cost of explosives and plant make a higher cost per cubic yard of material moved; and so on. The significance of all these conditions to the estimator will be appreciated by even the novice when he reads that the cost of excavation per cubic yard runs from below 10 cents to upward of $5, according to the circumstances of the work.

92. Estimating Data Should Be Prepared in Advance.—Many of the data for estimating costs are taken, as we have seen, from experience. It is not sufficient to have on file the pay-rolls and receipted bills for one or more past pieces of work of character similar to that of the one for which the estimate is being made. It is seldom practicable to take the time to extract the useful data from a great mass of original records of construction at the time of preparing a preliminary estimate, and as a result many an estimate is made in an office stored with original records which, though filled with potentially usable data, lend little or no aid to the estimator. The original records should be sorted, sifted, and studied as soon as made, or as soon thereafter as practicable; selected data should be tabulated; and ratios, relations, and laws, which may be of direct use in making preliminary estimates, should be derived from the data and recorded in the shape of formulas, or tables, or charts, all of which should be accompanied by de-

scriptions of all pertinent circumstances and conditions of the work from which the data are drawn.

In short, the data of experience should be reduced to their lowest or most convenient terms at the earliest opportunity. The foregoing may be reinforced by the following, taken from "De Pontibus,"[1] pp. 346 and 347:

"In concluding this chapter on 'Office Practice' the author desires to call again the reader's attention to the necessity for adopting the most systematic methods possible for doing all kinds of work, keeping all kinds of records, and filing all kinds of accumulated material. As soon as a large piece of work is finished a thorough systemization should be made of the knowledge obtained in making both the design and the various calculations, so that the office force shall be able to use the same to the best possible advantage when starting on another piece of work. And whenever there is any spare time in the office for any of the employees it should be devoted to accumulating, digesting, and putting in convenient form for use the results of previous investigations, and to doing such work as tabulating and recording on diagrams the weights of metal per lineal foot of span for bridges of all kinds.

"Finally, in bringing this little treatise to a close the author feels that he cannot do better than repeat from Chapter II the following principle: 'The systemization of all that one does in connection with his professional work is one of the most important steps that can be taken toward the attainment of success.'"

(These words of Dr. Waddell will bear repeated reading by all engineers who have not already taken them to heart.)

93. References to Discussions on Estimating Cost of Proposed Structures.—A general idea of the possibilities and limitations which are characteristic of the work of estimating the cost of proposed structures may be obtained by reading the following articles:

In *Engineering and Contracting:*

Cost analysis in engineering colleges, vol. 28, p. 347;
Abuse of those who use cost data, vol. 28, p. 339;
Ashokan controversy, vol. 29, pp. 77, 95, 99–102, 111–114, 127.

In *Engineering Record:*

Cost of engineering work, vol. 56, p. 584;
Abuse of cost data, vol. 56, p. 611;
Value of cost data, vol. 56, p. 694;
Comparing estimates with construction costs, vol. 56, p. 136;

[1] "De Pontibus, a Pocket Book for Bridge Engineers," by J. A. L. Waddell; New York, John Wiley & Sons.

The estimation of construction expenses, vol. 56, p. 169;

The estimated cost of the main Ashokan dams: a statement of the methods followed by the engineers of the Board of Water Supply, vol. 57, pp. 181–184.

In *Engineering News:*

Ashokan contract award, vol. 59, pp. 17, 20, 107;

Additional discussion of this subject will be found in Gillette's "Cost Data," 2d ed., pp. 41–48.

B. Effect of Laws of Error on Estimates

94. Simple and Compound Estimates.—When we estimate the area of a field by simply looking at it and without consciously estimating the dimensions, the result is a "simple estimate" of the area. But if we estimate the length or breadth or both, and obtain the area by taking the product of the dimensions, the result is a "compound estimate" of the area. A compound estimate is a quantity derived from one or more quantities which have been estimated.

95. Absolute and Relative Errors.—Let A be the true, and a the estimated value of a quantity; let e be the absolute and r the relative error of a. Then the absolute error is

$$e = a - A; \tag{59}$$

and the relative error is

$$r = (a - A)/A = e/A; \tag{59a}$$

or, approximately,

$$r = e/a. \tag{59b}$$

Example.—A distance is estimated to be 250 feet. Later it is found, by careful measurement, that the distance is 263.2 feet. What is the absolute error of the estimate? What is the relative error? The absolute error is $e = a - A = 250 - 263.2 = -13.2$ feet. The relative error is $r = e/A = -13.2/263.2 = -0.0502$; or approximately, $r = e/a = -13.2/250 = -0.0528$. For all practical purposes the approximate relative error is as serviceable as the true. In fact in this case it would be just as well to say "the relative error is -5 per cent."

96. Limits of Error.—*Limit of Error in Simple Estimate.*—The engineer is not satisfied to accept an estimate, whether made by himself or another, without knowing something of its quality.

He wants to know about how far the true value may lie to one side or the other of the estimate.

Until we obtain the true value of a quantity of which we have an estimate, we must look to past experience for information concerning the limit of error in the estimate. A man can acquire, in a few years, from those comparisons of estimated and measured quantities which are thrust upon him, a fair knowledge of the limit of error in his estimate of a given kind of quantity under given conditions; but he can acquire the same knowledge in a much shorter time if he will take the extra trouble of estimating every quantity which he is about to measure, and noting the sign and amount of the error discovered by the measurement.

Limit of Error in a Compound Estimate.—We shall consider here only the four simplest cases, namely, those in which the compound estimate is derived from simple estimates by addition, subtraction, multiplication, and division, respectively.

Formulas for Limits of Absolute and Relative Error of Compound Estimates

Compound estimate formed by	Limit of absolute error	Limit of relative error
Addition: $a_1+a_2+\ldots+a_n$	$\pm(e_1+e_2+\ldots+e_n)$ (60)	$\pm\dfrac{e_1+e_2+\ldots+e_n}{a_1+a_2+\ldots+a_n}$ (60a) or, $\pm\dfrac{a_1r_1+a_2r_2+\ldots+a_nr_n}{a_1+a_2+\ldots+a_n}$. (60b)
Subtraction: a_1-a_2	$\pm(e_1+e_2)$ (61)	$\pm\dfrac{e_1+e_2}{a_1-a_2}$. . . . . . (61a) or, $\pm\dfrac{a_1r_1+a_2r_2}{a_1-a_2}$. . . (61b)
Multiplication: $a_1\cdot a_2\cdot\ldots\cdot a_n$	. .	$\pm[(e_1/a_1)+(e_2/a_2)+\ldots+(e_n/a_n)]$ . . (approx.) (62a) or, $\pm(r_1+r_2+\ldots+r_n)$. . (approx.) (62b)
Division: a_1/a_2	. . . .	$\pm[(e_1/a_1)+(e_2/a_2)]$. . . (approx.) (63a) or, $\pm(r_1+r_2)$. . . (approx.) (63b)

In the formulas above, let

$a_1, a_2, \ldots, a_n$ = estimated values of quantities;
$e_1, e_2, \ldots, e_n$ = limits of absolute error in $a_1, a_2, \ldots, a_n$;
$r_1, r_2, \ldots, r_n$ = limits of relative error in $a_1, a_2, \ldots, a_n$.

The formulas for the four cases are given here in tabular arrangement, without the details of derivation.

97. Compensation of Errors.—An estimate may be wide of the truth because of some mistake, blunder, or bias; or because of some misapprehension or illusion, of which latter the following is a familiar and typical example: The road running before us slopes upward toward the foot of the mountain, but appears to be level or even to slope downward. All estimates of slope of such a road by men looking along it toward the mountain will be in error with the same sign. But passing beyond all such cases, experience has shown that in the long run positive and negative errors occur with equal frequency, and larger errors occur less frequently than smaller ones. For example, let the probability be 1/2 that a simple estimate will be in error by more than a given percentage of the limiting error; then the probability that the errors in two simple estimates will each be greater than the given percentage of the limiting error is 1/4; and that the two errors will also be of like sign is 1/8; the probability that the error in each of three estimates will be of like sign and greater than the given percentage of the limiting error is 1/32; and so on.

Thus we see that, while the error in a simple estimate may be very small and is more likely to be small than large, the probability of its being near the limit is great enough to make it prudent to be prepared to find it there; but, due to the neutralizing effect of positive and negative errors, the probability that the error in a compound estimate is near its limiting size grows rapidly less as the number of simple estimates which enter therein is increased.

98. The Working Principle.—The practical conclusion to be drawn from the facts presented in the foregoing section is this:

A compound estimate of a whole, formed by summing the estimates of the parts, is more reliable than a simple estimate of the whole (provided, of course, that the estimator have the same knowledge of parts as he has of the whole).

And from this principle follows another:

The greater the number of parts into which a whole is divided for the purpose of estimate, the greater is the reliability of the compound estimate of the whole.

For example, supposing that the builder be equally intimate with whole buildings and parts of buildings, his estimate of the cost of a proposed building would be more reliable if made by the "method of parts" (§88) than if made by the "method of total cost" (§86); and if made by the "method of complete analysis" (§90), more reliable than if made by the said "method of parts" (§88).

But the compensation of errors is not the only, and under some conditions may be the least, advantage which comes from estimating a whole by parts. When the estimator divides a whole into parts for the purpose of estimate, the lines are drawn, consciously or unconsciously, in such a way as to permit his taking advantage, as far as possible, of whatever special skill he may have in estimating certain kinds of quantities. For example, in the case of a house the estimator, if a mason, will probably part the masonry from the rest whether he subdivides the rest or not; if a plumber, segregate the plumbing; and so on.

Let the whole be divided into unlike parts. Let each part be estimated in subdivisions by an estimator who is specially qualified to estimate that part. Add together all the estimates. This is the recipe for making the most reliable estimate of the whole.

C. On Estimating Quantities of Some Particular Types

99. Future Rate of Output of Labor.—*One Laborer.*—How many days will one laborer require to load 300 cubic yards of sand? The volume of sand which a man will shovel into a wagon in one day depends on the mental and physical make-up and condition of the man, the condition of the sand, and the general conditions under which the shoveling is done. Every one of these factors is variable, and the best we can do is to find a maximum output, for one man-day, which can be reasonably hoped for and a minimum which we should be prepared for, and place our estimate inside these limits, either half way or nearer to one limit than to the other, according to our judgment based on a knowledge of the case. For example, experience shows, let us say, that a strong man in good condition of mind and body will shovel, under favorable working conditions, about 25 cubic yards of sand a day; while a weak man in low state of mind and body will shovel, under unfavorable conditions, about 15 cubic yards a day. If this be true, we know that if all factors happen to be favorable it

will take one man about 300/25 = 12 days to shovel 300 cubic yards; and if unfavorable, about 300/15 = 20 days. However, it is hardly to be expected that for 12 days on end a man will be at top-notch mentally and physically and for the same 12 days the sun will shine and all other conditions will favor the work; nor is it to be expected that a man will be in his lowest mental and physical states and the rain will fall and all other conditions will combine to lower the output for 20 days together. Even the distance through which the sand is shoveled must change during the progress of the work. On the other hand some factors, like the relation between supply and demand for shovelers, may exert a constant influence on the output. Therefore it is reasonable to expect one man to shovel the 300 cubic yards of sand in somewhat more than 12 days and in somewhat less than 20 days.

Group of Laborers.—The foregoing observations hold for each man of a gang of shovelers; but owing to the fact that *all* the men in a gang are neither of the most inefficient type and lowest physical and mental condition nor of the most efficient type and in highest physical and mental condition, the per cent. difference between the highest and lowest daily outputs is much less than for an individual. On the other hand, the difference between the highest and lowest daily outputs of a given number of men working in a gang will probably be greater than that of the same men working independently, because in the former case a greater number of conditions affect the total result.

100. Future Rate of Output of a Machine.—The average daily, monthly, or yearly output of a machine, like that of a laborer, depends on many variable conditions. A pumping engine in connection with a standpipe or reservoir may be run almost continuously under nearly uniform conditions, provided the demand for water be sufficient. Such an engine has the advantage of a steady job, and its maximum and minimum average rates of output are not far apart. Typical of structures which operate under the other extreme of conditions is the plant of the grading contractor. The potential capacity of, say, a steam-shovel can be computed from the dimensions of the dipper, time of round trip of dipper, time required to move up, and area of breast of cut. But the shovel can add to its output only when there are empty cars within its reach; and many circumstances interfere with keeping the shovel continuously supplied with these. Besides the interruptions to actual operation, due to more or less frequent

lack of cars, there are the interruptions caused by the necessity of backing up, and of moving from cut to cut; and, finally, the intervals, some long, some short, between jobs. There is a wide range between the output under the most favorable conditions which can be reasonably hoped for and that under the most unfavorable conditions which reasonably ought to be prepared for.

So in making an estimate of the future average rate of output of a structure, we must consider not only the conditions which may have an influence on the rate of output while the structure is actually working, but also the conditions which determine the proportion of working time to the time of enforced idleness.

101. Life of Proposed Structure.—The engineer can usually predict with confidence some of the general conditions under which the proposed structure will be operated. Studying past experience with similar structures operating under similar conditions, he will be able to fix upon the high and low limits of life between which the life of the proposed structure may be reasonably expected to fall. Judgment must tell him just what length of life, between these limits, to adopt.

It should be remembered that structure life may be terminated before the structure is worn out: by some change in the requirements of the service, as where a steel bridge is replaced, long before it is worn out, by a heavier one because of the increased weight of trains; by improvements, discoveries, or inventions, or by change in the demand for service, which make it economically necessary to replace the structure before it is worn out, as where the electric car replaces the horse car; or by accident.

Data on structure life will be found in Appendix C.

102. The Period of Service.—The period of service is one of the data necessary for the solution of the problem of economic selection. In some cases the period of service proposed may be definitely fixed; in others, uncertain. When the period of service is not otherwise determined it must be either estimated or assumed. Of course the same period of service must be used for all the structures which are under comparison at one time for one service. The following points deserve consideration.

The service required may increase with time. The railroad bridge of 30 years ago was designed for loads somewhat in excess of those then carried. Train weights were increased from time to time with the result that before the bridge was worn out it became too light for the changed requirement of the service. It

will be accepted without computation that it is unwise to build a structure today just adequate to today's demand if tomorrow's demand will be greater. On the other hand, a glance at a compound interest table (Table A) will show that it is not wise to build today for that increment of service which will not be in demand before, say, a hundred years hence. If, then, the requirements of service may be expected to increase with time, the period of service which is adopted as a basis of economic comparison of structures must be a compromise.

If the demand for a service continue beyond the life of the selected structure, at the end of that life the problem of economic selection of structure will again arise. For example, suppose a wooden highway bridge is wisely selected for a stated service, and built and operated. At the end of the life of this bridge the question as to whether the wooden bridge shall be renewed, or replaced by one of different weight or material, will call for answer. If at that time all the conditions and the data be the same as at present, the answer will be ready to hand: the wooden bridge should be renewed; but if some of the conditions or data at that time be different, the problem of economic selection must be solved anew.

103. Future Unit Prices.—The first cost of a structure which has been completed may be ascertained accurately from the record of expenditures, if such record have been faithfully made and preserved; but if such record be incomplete or missing, we must be content with an approximate estimate of the cost, based on such data concerning the quantities and unit prices of the time of construction as can be collected.

The estimate of cost of a proposed structure is necessarily based upon the unit prices of the future. Future prices are sometimes fixed in advance by agreement. For example, a contractor who purposes to bid on the construction of a pipe sewer obtains from some dealer an option on such pipe as is required; and is thus able to know beforehand just what unit prices he will have to pay for pipe in case his bid be accepted. Wherever such options are taken the exact prices are known in advance; but except in those cases where it is practicable thus to fix it in advance by option, a future unit price must be predicted or estimated.

Any prediction of a future unit price, as of a day's labor, must be based on the prices past and present, on the known variations therein, on our knowledge of the causes of the variations, and on

our forecast of conditions. If we plot the curve of unit price of labor for one locality over a long period, we find the curve to be sinuous. The curve rises and falls, but rises in the long run more than it falls. It is said that the price of labor depends on the ratio of supply and demand; that when the demand is strong and supply weak, price is high; and that when demand is weak and supply strong, price is low. This law is helpful in estimating future prices only in so far as we can foresee the future relation between supply and demand. If the curve of unit price of labor, just mentioned, run back over a period sufficiently long, we can draw an averaging line by eye and note its general characteristics and the extreme variations of the humps and sags from it. Then by projecting this averaging line into the future we obtain the tendency of average price in the future; and from the deviations of the sags and humps from the averaging line of the past, we can predict the probable deviations of the future. The distance into the future to which the price curve can be safely run, depends on the length of the past price curve, its characteristics, and our knowledge of the causes of past fluctuations and our ability to foresee the causes of future fluctuations.

The unit price, present or past, of any kind of material which is to be used in building a structure, may be ascertained in one or more of the following ways: (1) from personal knowledge; (2) from consultation with a dealer; (3) from consultation with a purchaser; (4) from catalogs and discount sheets; (5) from the "Monthly Price List of the Materials Used in Construction" to be found in the first issue of each month of *Engineering and Contracting* and *Engineering News;* (6) from Gillette's "Cost Data," 2d edition; (7) from Trautwine's "Civil Engineer's Pocket Book"; (8) from articles and papers in engineering periodicals and publications of engineering societies. See Appendix D for list of books which give cost data.

The cost of transportation to be furnished by a transportation company is in any case computed from rates supplied by an agent of the company.

104. Present and Future Population.—Structures are planned to meet future demands for service. Demand for service is usually a function of population of the territory served; and, other things equal, will increase when population increases. Hence, the first step in the determination of the demand for a proposed service, to be expected in a given territory during the coming

year, the second, the tenth, or the twenty-fifth year hence, is to estimate the population for that year.

Common Method.—A method of predicting future population, which at once suggests itself to the engineer, is to plot a curve of past populations, using years as abscissas and corresponding populations as ordinates; and extend the curve into the future. In this way the population for a given future year is read off the curve at its intersection with the ordinate of that year. However, the reliability of the estimate grows rapidly less as we advance along the continued portion of the curve; for as we advance, the greater becomes the probability that some unforeseen change in conditions will greatly modify the past variation in rate of growth.

Kuichling's Method.—Mr. Emil Kuichling, in seeking the law of growth of population, plotted such curves for several cities. Owing to the great diversity among the curves he could make no satisfactory deductions from them. He then found for each city and each decade the absolute increase for the decade; divided this absolute increase by ten, obtaining the average yearly increase; found the per cent. ratio of this average yearly increase to the population at the beginning of the decade; and plotted this per cent. ratio as an ordinate, using as an abscissa the population at the beginning of the decade. Thus "a new series of curves was obtained which exhibited much less divergence, and afforded a clue to the formulation of the law which seems to govern the growth of cities." He finally averaged the percentages which corresponded to the same population, and, using the average percentages for ordinates and corresponding populations for abscissas, plotted a new curve which expressed the law of growth of such cities. This was done in 1889, and at that time it was predicted by the use of this law that Rochester would have a population of 204,500 in 1900 and of 283,459 in 1910. For the former year the Census gives the population of that city as 162,608 and for the latter year 218,149. Thus time proved the estimate for 1900 to be about 25 per cent. too large and the estimate for 1910 to be about 30 per cent. too large. These results are given to show that even the most intelligently and carefully prepared predictions of future population may be considerably wide of the mark, and this when the prediction extends no farther into the future than 10 or 20 years. (See "Report on the Proposed Trunk Sewer for the East Side of the City of Rochester, N. Y., Made by Emil Kuichling, Civil Engineer, to the Common

Council, April 29, 1889," Appendix I: "The Growth of Population of American Cities of Medium Size," and Appendix II: "The Density of Population in Cities.")

Five Other Methods.—In Bulletin No. 135 of the Twelfth Census of the United States, devoted to "Methods of Estimating Population," Mr. Walter F. Wilcox, (then) Chief Statistician for Methods and Results, presents the results of his study of five methods of estimating population. The following quotations are from his work.

Method I.—"An estimate reached by assuming that the rate of growth between any two successive censuses, is maintained during the following decade." (Note that *rate of growth* and *increase of population* do not mean the same thing.) ". . . In half of the seventy-eight cities examined the per cent. of growth between 1890 and 1900 differed by eighteen or more from the per cent. of growth between 1880 and 1890. Therefore, the assumption that under present conditions the rate of growth of a given city tends to remain the same is inadmissible."

Method II.—"An estimate based upon the number of votes cast at an election." . . . "The number of votes cast at an election in a large city . . . stands in no constant or uniform relation to the population. In Albany, Columbus, and Dayton there were less than four inhabitants to each vote cast at the presidential election of 1900, while in several Northern cities there were more than eight and in certain Southern cities more than twelve to each vote cast. In the states and territories the increase in population runs by no means parallel with the increase of votes. Even when the Southern states are disregarded, as having few large cities, and exceptional conditions affecting the number of votes, still the per cent. of gain in the vote for the decade is likely to be as much as 8 greater or less than the corresponding per cent. for population."

Method III.—"An estimate based upon a school census." "The number of children of school age is a more accurate index of total population than prior rate of increase or vote cast. This method gives estimates of population half of which fall within 6 per cent. of the truth. But the number of children of school age in a city is so seldom given with close accuracy by a school census, that this method is found of little practical value."

Method IV.—"An estimate based upon a directory canvass." "The ordinary method of estimating the population of a large

city, that based on the number of names in the city directory, results uniformly in too large a figure and usually in very serious inaccuracy."

Method V.—"An estimate based upon the assumption that the absolute yearly growth is constant." "A method of estimating urban growth under present American conditions, which is certainly simpler and probably more accurate than any of the four here examined (I, II, III, and IV), may be commended to the consideration of interested city officials. This method is to add for each year after 1900 one-tenth of the city's increase from 1890 to 1900. It is merely a rule of thumb and without rational justification. But between 1880 and 1890 the seventy-eight cities together increased in population by an average annual amount of 407,028, and between 1890 and 1900 by an average annual amount of 414,793. As the rate of increase for the cities collectively fell from 46.8 per cent. between 1880 and 1890, but the total amount of increase between 1890 and 1900 exceeded by less than 2 per cent. the total amount between 1880 and 1890, it is evident that the assumption of a constant amount is much nearer to the truth than the assumption of a constant rate. If this method had been applied to the cities for 1900, the result in half the cases would have been within 6 per cent. of the truth, closer than estimates based upon votes cast or number of names in the directory, and as close as the estimates resulting from an accurate census of school children, were that obtainable. From the point of view of local authorities, however, it may be an objection to this simple method that, on the basis of the estimates for 1900, it is as likely to result in an underestimate as in an overestimate, while the method now most in vogue, that based on the number of names in the city directory, produces nearly always an overestimate."

"It is not the intention of this bulletin to criticise the use of these methods [I, II, III, and IV] where nothing better can be secured; the aim has been merely to give the reasons of the Census Office for doubting that the results of such methods are entitled to serious consideration when they contradict those of a careful enumeration."

105. Future Earnings.—An estimate of future earnings may be based upon past earnings either of the same structure or of similar structures operated under more or less similar conditions. For example, to estimate the earnings of a proposed street rail-

way in a given town not yet provided with a street railway, we should obtain and study the statistics of earnings of street railways already operating in towns more or less similar in size and character to the given town, and seek to find some law directly usable for our estimate; as, for instance, the ratio of earnings to population. By plotting a curve of population (§104) and a curve of earnings for the same town, we have presented to the eye the general relation between population and earnings, for the conditions there obtaining. These conditions include the character of the population, its distribution and occupation; the location of the line with respect to the centers of residence and business districts; and so on. Plotting sets of curves for as many typical towns as data can be obtained for, we are in a position to discover the ratios between earnings and population for a range of conditions. Applying such of these ratios as appear to be applicable to the estimated population year by year of the town for which the railway is proposed, we obtain the most reliable estimate possible of the earnings year by year of the proposed railway.

The subject of estimating future earnings is treated at length in the following books:

"Economics of Railroad Construction," W. L. Webb, New York, John Wiley & Sons, second edition, 1912, 339 pages, $2.50. (Chapter V is on the estimation of volume of traffic.)

"Economic Theory of Railway Location," A. M. Wellington, New York, John Wiley & Sons, 1886, 950 pages, $5. (Chapter IV is on the probable volume of traffic and the law of growth therein.)

"Electric Railway Economics," W. C. Gotshall, New York, McGraw-Hill Book Co., 1903, 251 pages, $2. (Chapter VI is on estimating earnings.)

PART III

SOLUTION OF THE PROBLEM OF ECONOMIC SELECTION

CHAPTER VIII

BASIS OF ECONOMIC COMPARISON

This chapter states the fundamental principle of comparison, and presents four bases of comparison. The working formulas are given in Table VII.

106. Fundamental Principle.—How does one set about selecting one of two or more structures proposed to perform a stated service? By estimating in turn the cost of the service which would result from the use of each of the proposed structures, and comparing the resulting costs. Cost of service has been considered in Chapter VI, but comparisons are still to be considered. Any legitimate basis of comparison of costs must be founded on the following principle which results from compound interest:

To compare the values of two or more payments made on different dates, the payments must be reduced to equivalent payments of some one date or to equivalent series of uniform payments of one set of dates.

Upon this principle rest the four bases of comparison which follow.

107. Basis A. Yearly Cost of Service.—The equivalent uniform yearly cost of service over the stated period is computed for each structure that enters into the comparison. For this computation some one of the following formulas is used.

The formula to be used for computing this yearly cost for a given structure will depend upon the ratio of the life (L years) of that particular structure to the length of the stated period of service. A formula has been written for each of these four conditions: (1) when the stated period of service is equal to a whole number of lives plus a fractional life (this formula is general, as it covers every possible case, but being clumsy for the following three cases it has been rewritten for each of them); (2) when the stated period of service is less than one life of the structure; (3)

when the stated period of service is equal to one or more whole lives of the structure; and (4) when the service is assumed to be continued forever.

The general formula for yearly cost of service (§72) is

yearly cost of service = yearly cost of amortization
+ yearly cost of interest and other fixed charges
+ yearly cost of operation
+ yearly cost of maintenance. (64)

Each of the following equations is formed from Eq. 64 by substituting for each of the four right-hand terms the proper symbolic expression which we find, ready to hand, in Chapter VI.

1. *When the Stated Period of Service is Equal to a Whole Number of Lives Plus a Fractional Life.*—The formula for this case is formed by substituting in Eq. 64 from Eqs. 53, 54, 55, and 58. Thus the yearly cost of service for the condition named is

$$y_{n'}=[(C-C_L)d_Lw_n+(C-C_A)d_A(w_{n'}-w_n)]a_{n'}+Cr+O+M \qquad (65)$$

where A = fractional life at end of period of service;
n = number of years in the whole number of lives;
n' = period of service;
$= n + A$;
L = life of structure;
C = first cost of structure;
C_A = salvage value at age A;
C_L = salvage value at end of life;
d_A = annual deposit in sinking fund to redeem $1 at the end of A years;
d_L = annual deposit in sinking fund to redeem $1 at the end of L years;
w_n = present worth of $1 per year for n years;
$w_{n'}$ = present worth of $1 per year for n' years;
$a_{n'}$ = annuity produced for n' years by $1 principal;
r = ratio of yearly interest and other fixed charges to first cost;
O = yearly cost of operation;
M = yearly cost of maintenance.

As remarked above, Eq. 65 can be used for any case but is inconvenient for the following cases.

2. *When the Stated Period of Service is Less Than One Life.*—

Substituting in Eq. 64 from Eqs. 51, 54, 55, and 58, we have the following expression for yearly cost of service in this case:

$$y_A = (C - C_A)d_A + Cr + O + M \tag{66}$$

where A = the period of service, A being less than L the life of the structure;
C = first cost of structure;
C_A = salvage value at age A;
d_A = annual deposit in sinking fund to redeem \$1 at the end of A years;
r = ratio of yearly interest and other fixed charges to first cost;
O = yearly cost of operation;
M = yearly cost of maintenance.

3. *When the Stated Period of Service is Equal to One or More Whole Lives.*—The formula for this case is formed by substituting in Eq. 64 from Eqs. 52, 54, 55, and 58. Thus for this case the yearly cost of service is

$$y_n = (C - C_L)d_L + Cr + O + M \tag{67}$$

where n = the period of service (n is a multiple of L);
L = life of structure;
C = first cost of structure;
C_L = salvage value of structure at the end of life;
d_L = annual deposit in sinking fund to redeem \$1 at the end of L years;
r = ratio between yearly interest and other fixed charges and first cost;
O = yearly cost of operation;
M = yearly cost of maintenance.

4. *When the Stated Period of Service is Perpetual.*—The yearly cost of service in this case is the same as in the preceding; and, using the subscript ∞ to indicate the length of the period of service here, Eq. 67 becomes

$$y_\infty = (C - C_L)d_L + Cr + O + M \tag{68}$$

$$= Cr + O + M \quad \text{(when } L \text{ is infinite)}, \tag{68a}$$

where the notation is the same as above.

The foregoing formulas for yearly cost of service are presented in Table VII, on the horizontal line beginning with A.

108. Basis B. The Capitalized Cost of Service.—Capitalized cost, as the term is used here, is present worth. Thus the capitalized cost of service performed by a structure is the sum of the present worths of all the expenditures that are necessary to the service—first cost and succeeding outlays. In the foregoing section we have presented formulas for computing the yearly cost of service in such a way that the series of yearly costs thus computed shall be the equivalent of the actual costs. Now the present worth of the actual costs is the same as the present worth of the equivalent series of yearly costs. Thus the capitalized cost of service can be obtained by multiplying the yearly cost, as computed by one of the formulas of the preceding section, by the present worth of $1 per year for the number of years in the stated period of service. This will explain the derivation of the four formulas which appear on line B of Table VII and give the capitalized cost of service corresponding to periods of service of $n'(= n + A)$ years, A years, n years, and an infinite number of years, respectively. For example, in column 2, line B, of Table VII is found the formula for capitalized cost of service for A years (less than the life of the structure), namely, $Y_A = y_A w_A$, where y_A has the value $(C - C_A)d_A + Cr + O + M$ as shown immediately above in column 2, and w_A is the present worth of $1 per year for A years.

109. Basis C. Cost per Unit of Service.—For each structure considered in the comparison, the cost per unit of service is obtained by dividing the yearly cost of service (of Basis A) by the yearly output, denoted by U. In Table VII, on the line beginning with C, are written the four formulas for computing cost per unit of service. For example, in column 2, on line C, is given the formula for cost per unit of service for the case in which the period of service is A years (less than one life), namely, $v_A = y_A/U$, where y_A has the value given it on line A, column 2.

110. Basis D. Capitalized Cost Per Unit of Service Per Year.—The capitalized cost per unit of service per year, for each of the structures under comparison for a stated service, is obtained by dividing the capitalized cost of service (Basis B) by the yearly output, U. The four formulas for computation on this basis (D) will be found on line D of Table VII.

CHAPTER IX

PROCEDURE FOR ECONOMIC SELECTION

This chapter gives in order the steps of the routine of economic comparison and selection. Illustrative examples are given in Chapter X.

111. The Procedure and Steps Therein.—The work of solving a problem of economic selection is here divided, for convenience, into six main divisions, which, in the following, are called "steps." The six steps are:

Step 1.—Collecting numerical data.
Step 2.—Collecting outstanding data.
Step 3.—Choosing a basis of comparison.
Step 4.—Choosing a formula.
Step 5.—Computing cost of service.
Step 6.—Making the decision.

The work cannot usually be made to follow strictly the sequence indicated by the steps. For example, it will be found that while some numerical data (step 1) are prerequisites to choosing a formula (step 4), the work of making complete the list of numerical data is greatly helped by reference to the selected formula.

For illustrative examples for each step turn to the next chapter (X).

112. Step 1. Collecting Numerical Data.—Collect *all* the pertinent numerical data for each of the structures under comparison; and list the data in an orderly manner to facilitate reference.

One of the data always pertinent is the period of service for which the comparison is to be made.

As soon as data sufficient for the purpose have been listed, choose a basis of comparison (step 3); and then for each structure choose a formula (step 4) for computing cost of service.

The formula chosen for each structure may materially aid in completing the list of data for the structure.

The schedules of the following sections may lend additional aid, by way of jogging the memory, in insuring the completeness of the data: §§39 and 40 (first cost), §§50, 52 and 60 (salvage value and depreciation formulas), and §72 (elements of cost of service).

Some of the listed data may be in form unsuitable for direct substitution in the chosen formula, and such data must be reduced to suitable form. This reduction of data may in some cases require far more computing than that indicated by the chosen formula. The reduction of data is mentioned as a preliminary operation in step 5.

113. Step 2. Collecting Outstanding Data.—List for each structure *all* those advantages and disadvantages which pertain to the structure but are not appraised in terms of money and are not recognized in the chosen formula (step 4).

Outstanding data include all of those data which the engineer cannot, or chooses not to, evaluate in terms of money, and which therefore he cannot, or does not, recognize in the formulas for cost of service. See §§47 and 84.

The following are typical of outstanding data, but the list should not be assumed to be complete.

1. Extra cost of operating during the construction period (§47).
2. Loss of net income due to interruptions to operation during construction.
3. Risk, during construction, of accident to operation or to property or persons of outside parties.
4. Risk, during construction, of accident which may protract the period of construction, or require a repetition of a part of the work or a duplication of a part of the material or construction plant, or bring about all these results, with the consequent lengthening of the construction period, besides direct extra outlay.
5. Ease or difficulty with which capital is obtained.
6. Risk of accident after construction, which may increase the cost of maintenance or of operation or of both, or decrease the output.
7. Risk of damage to outside property or persons due to operating the structure.
8. The outlook for the prompt and suitable supply of labor and materials for construction, and for operation and maintenance.

114. Step 3. Choosing a Basis of Comparison.—Choose one of the following four bases of comparison for each problem of economic selection:

Basis A, yearly cost of service (§107 and Table VII).
Basis B, capitalized cost of service (§108 and Table VII).
Basis C, cost per unit of service (§109 and Table VII).

Basis D, capitalized cost per unit of service per year (§110 and Table VII).

If the structures under comparison have *equal* outputs, all four of the bases named are equivalent one to another, and personal preference will dictate which one of the four shall be used.

If the structures under comparison have *unequal* outputs, it is correct to use bases C and D only, and the choice between the two is a matter of individual preference.

115. Step 4. Choosing a Formula.—Select for each structure the appropriate formula from Table VII.

The formula to be used for a given structure depends (1) on the basis of comparison chosen in step 3, and (2) on the relation between the life, L, of the structure and the stated period of service.

Table VII is arranged as follows: on each horizontal line is written a basis of comparison followed by all the formulas which are proper to that basis. Column 1 contains all the formulas which are applicable whatever may be the period of service and life of structure.

One of the formulas of column 1 must be used for the case in which the period of service is equal to one or more whole lives plus a fractional life. (For most cases the formulas of column 1, which are general, reduce to simpler forms which are given in succeeding columns.) For example, if the basis of comparison be "capitalized cost per unit of service per year" and the stated period of service be n' years (= one or more whole lives covering altogether n years + a part of one life = $n + A$ years), the proper formula will be Eq. D1 of Table VII:

$$V_{n'} = y_{n'}w_{n'}/U.$$

If the life of the structure be greater than the stated period of service, use one of the formulas of column 2. For example, if the basis of comparison be "yearly cost of service" and the period of service be A years (A being less than L the life of the structure), the proper formula will be Eq. A2 of Table VII:

$$y_A = (C - C_A)d_A + Cr + O + M.$$

If the period of service be equal to one or more whole lives use one of the formulas of column 3. For example, if the basis of comparison be "capitalized cost of service" and the stated period

of service be n years (n being equal to L, or $2L$, or $3L$, or . . .), the proper formula will be Eq. B3 of Table VII:

$$Y_n = y_n w_n.$$

If the period of service be taken as of infinite duration, use one of the formulas of column 4. For example, if the basis of comparison be "cost per unit of service" and the stated period of service be infinite, the proper formula will be Eq. C4 of Table VII:

$$v_\infty = y_\infty / U.$$

116. Step 5. Computing Cost of Service.—The cost of service corresponding to each structure is found by substituting the numerical data of step 1 in the formula chosen in step 4, and making the indicated computation.

If any of the data of step 1 are not already in form suitable for immediate substitution in the formula, such data must be reduced to suitable form as a preliminary to such substitution.

The work of computing is facilitated by arranging, as far as practicable, the arithmetic operations and results in some tabular form. (See the following chapter (X).)

The most economical structure, *so far as can be determined from the numerical data,* is that one for which the computations just described give the lowest result; but the result of the computation may not decide the matter of selection if there be any *outstanding data* (step 2).

117. Step 6. Making the Decision.—Having made sure that *all* data—numerical and non-numerical—which bear directly or indirectly on the question, have been collected; having assigned the most accurate values to the numerical data, and proper weights to the remaining data; having selected a suitable basis of comparison and the proper formula; having computed the cost of service by the formula, and insured the correctness of the mathematical work—in short, having logically and accurately compressed all the pertinent data into the most readily comparable form, the engineer finally bases his decision on a comparison in which he weighs well both the computed costs of service *and* the outstanding data.

Not infrequently in practice the difference in weights of outstanding data pertaining to two structures is so greatly to the advantage of one of them as to overshadow any advantage which the difference in computed costs shows for the other. For example, the cost of service may be correctly computed to be de-

cidedly less for a structure of high first cost than for another of low first cost; outstanding data pertaining to the structure of low first cost may nearly all be to its further disadvantage; yet if, among the data for the structure of high first cost, we find "first cost exceeds the capital available for the purpose," the choice must fall on the structure of low first cost.

CHAPTER X

EXAMPLES OF ECONOMIC SELECTION

In this chapter will be found fifteen numerical examples of economic comparison and selection, worked out in such detail as to serve as models of form and procedure.

118. Introduction.—In the following examples all computations made in estimating first cost, operating cost, etc., have been omitted. We are concerned in this chapter solely with the process of economic selection.

The 10-inch slide rule will give cost of service of a proposed structure with all the accuracy usually possessed by the data which enter into the computations. The only reason for giving results to the nearest cent in this chapter was to secure consistency among the quantities which appear on the printed page.

The arrangement of the computations in this chapter has been influenced by the limited width of the printed page and the need of interjecting a reference or an explanation from time to time. For office use the following computing form has some advantages: the arrangement is compact as well as clear; only one column of symbols is needed; and the resulting costs of service stand side by side, making comparison easy.

Suggested Computing Form for Office Use

Symbols	Structure No. 1		Structure No. 2		Structure No. 3	
		$		$		$
C..........						
C_L..........						
$C - C_L$..........						
d_L..........						
$(C - C_L)d_L$..........						
....						
....						
....						
....						
C_r..........						
O..........						
M..........						

It will be noted that in some of the following examples the numerical data are so overwhelmingly in favor of one of the structures proposed that no formal computation is needed to show which one of the structures is the most economical. In such a case it is still necessary to compute the numerical advantage in order accurately to weigh it with the outstanding data. One such case is given in §131.

119. Wooden Bridge v. Steel Bridge.—Required to select a bridge as for a wagon road to a lumber camp of which the estimated life is eight years.

Step 1. *Numerical Data.*—The period of service is eight years.

	Wood	Steel
First cost..	$280	$400
Life (years)	12	30
Salvage value at age of eight years	0	$200
Rate of interest on amortization fund, per cent	5	5
Rate of interest on capital, per cent.	5	5
Tax rate, per cent	½	½
Yearly cost of operation	0	0
Yearly cost of maintenance..	$5	$5

Step 2. *Outstanding Data.*—The wooden bridge can be made ready for use three months sooner than the steel.

Step 3. *Basis of Comparison.*—The quantity of service will be the same whether the one or the other bridge is built, but since it is not given we are limited to bases A and B, of Table VII. Let us use basis A as requiring less work.

Step 4. *The Formula.*—Since we are to use basis A and the period of service is eight years, which is less than the life of either of the proposed bridges, the proper formula is A2 of Table VII.

Step 5. *Computation.*—The formula, A2 of Table VII, for yearly cost of service, is

$$y_A = (C - C_A)d_A + Cr + O + M.$$

For the wooden bridge:

C.. 280
C_A............ 0
$C - C_A$........ 280
d_A...........0.1047 (opposite "8," under "5 per cent.," Table E)

$(C - C_A)d_A$		\$29.32	
Interest rate	0.05		
Tax rate	0.005		
r	0.055		
Cr		15.40	
O		0.00	
M		5.00	
y_A = yearly cost of wooden bridge =			\$49.72

For the steel bridge:

C	\$400
C_A	200
$C - C_A$	200
d_A	0.1047 (opposite "8," under "5 per cent.," Table E)

$(C - C_A)d_A$		\$20.94	
Interest rate	0.05		
Tax rate	0.005		
r	0 055		
Cr		22.00	
O		0.00	
M		5.00	
y_A = yearly cost of steel bridge =			47.94
Difference in yearly costs			\$ 1.78

Step 6. *The Decision.*—The yearly cost of the steel bridge is \$1.78 less than that of the wooden bridge. However, being able to use the wooden bridge three months earlier than the steel is considered to be worth more than \$1.78 per year; and the wooden bridge is chosen.

(If we had used basis B instead of basis A, the formula would have been B2 of Table VII: $Y_A = y_A w_A$; and the computations above for y_A would have been carried out just as they have been. The value of w_A would then have been found in Table D, opposite "8" and under "5 per cent.," to be 6.463, and we should have found that for the wooden bridge

$Y_A = 49.72 \times 6.463 =$ $321.34

and for the steel bridge

$Y_A = 47.94 \times 6.463 =$ 309.84

The difference in capitalized costs is $11.50

and this is to the advantage of the steel bridge. (The same result may be obtained by capitalizing the yearly advantage, $1.78, thus: 1.78 × 6.463 = $11.50.) Thus, the advantage of the earlier use of the wooden bridge is considered to be worth more than $11.50 in hand.)

120. Wooden Bleachers v. Concrete on Earth Bank.—These two types of structure were proposed for a given service. Which is the more economical of the two on the assumption that the service will be required for 90 years?

Step 1. *Numerical Data.*—The period of service is 90 years.

	Concrete	Wood
First cost	$80,000	$30,000
Life (years) . . .	90	15
Salvage value at end of life . .	0	$2,000
Interest rate on amortization fund, per cent	6	6
Interest rate on capital, per cent.	6	6
Insurance rate, per cent .	0	0.7
Yearly cost of operation .	0	0
Yearly cost of maintenance	$100	$300

Step 2. *Outstanding Data.*—None.

Step 3. *Basis of Comparison.*—Capitalized cost of service was taken as the basis. This is basis B of Table VII.

Step 4. *Formula.*—Since the chosen basis is B, the proper formula for each structure will be found on line B of Table VII. The period of service is a multiple of the life of the wooden structure, also of the life of the concrete and earth structure. Therefore the proper formula is B3 of Table VII.

Step 5. *Computation.*—The formula, B3 of Table VII, for capitalized cost of service, is

$$Y_n = y_n w_n$$

where

$$y_n = (C - C_L)d_L + Cr + O + M. \qquad \text{(A3)}$$

For the concrete structure:

Eq. B3 becomes

$$Y_{90} = [(C - C_{90})d_{90} + Cr + O + M]w_{90}.$$

C	80,000	
C_{90}	0	
$C - C_{90}$	80,000	
d_{90}	0.0003	(opposite "90," under "6 per cent.," Table E)

$(C - C_{90})d_{90}$		\$24.00
Interest rate	0.06	
Insurance rate.	0.00	
r	0.06	
Cr		4800.00
O		0.00
M		100.00
y_{90} = yearly cost for concrete structure =		\$4924.00
w_{90}	16.58	(opposite "90," under "6 per cent.," Table D)

$Y_{90} = y_{90}w_{90}$ = capitalized cost for concrete structure \$81,639.92

For the wooden structure:
Eq. B3 becomes

$$Y_{90} = [(C - C_{15})d_{15} + Cr + O + M]w_{90}.$$

C	30,000	
C_{15}	2,000	
$C - C_{15}$	28,000	
d_{15}	0.0430	(opposite "15," under "6 per cent.," Table E)

$(C - C_{15})d_{15}$		\$1204.00
Interest	0.06	
Insurance	0.007	
r	0.067	
Cr		2010.00
O		0.00
M		300.00
y_{90} = yearly cost for wooden structure		\$3514.00
w_{90}	16.58	(opposite "90," under "6 per cent.," Table D)

$Y_{90} = y_{90}w_{90}$ = capitalized cost for wooden structure 58,262.12

Difference between capitalized costs.............. \$ 23,377.80

Step 6. *The Decision.*—The capitalized cost of the wooden structure is less than that of the concrete by \$23,377.80, and since there is no outstanding datum (step 2) the choice must fall on the wooden structure.

121. Economic Size of Pipe-line for Uniform Delivery.—A mining company wishes to know which one of its stock sizes of pipe will be the most economical for a proposed pipe-line through which to pump water under the conditions given below.

Step 1. *Numerical Data.*—The period of service is five years, and during this time the flow is to be regularly 1 cubic foot per second. The length of pipe-line is taken as 1 mile.

Stock sizes of pipe	6 inches	8 inches	10 inches
First cost of 1 mile of pipe laid	\$5000	\$7000	\$9200
Salvage value at the age of five years (25 per cent.)	\$1250	\$1750	\$2300
Interest rate on amortization fund, per cent.	6	6	6
Interest rate on capital, per cent	6	6	6
Tax rate, per cent.	1	1	1
Yearly cost of operation	\$1740	\$414	\$146
Yearly cost of maintenance	0	0	0

Step 2. *Outstanding Data.*—There is some probability that the demand will be 2 cubic feet per second during the last two years.

Step 3. *Basis of Comparison.*—A convenient basis of comparison in this case is the total yearly cost of the service rendered by the pipe-line, and this basis is A of Table VII.

Step 4. *Formula.*—With a period of service of five years, which is less than the life of pipe, and basis A, the proper formula is A2 of Table VII.

Step 5. *Computation.*—The formula, A2 of Table VII, for yearly cost of service, is

$$y_A = (C - C_A)d_A + Cr + O + M.$$

For the 6-inch pipe:

C.......... 5000
C_A.......... 1250
$C - C_A$..... 3750
d_A.......... 0.1774 (opposite "5," under "6 per cent.," Table E)
$(C - C_A)d_A$............................ \$665.25
Interest rate... 0.06
Tax rate...... 0.01
r............ 0.07

Brought forward	665 25	
Cr	350.00	
O	1740 00	
M	0.00	
y_A = yearly cost with 6-inch pipe		\$2755.25

For the 8-inch pipe:

C	7000	
C_A	1750	
$C - C_A$	5250	
d_A	0.1774 (opposite "5," under "6 per cent.," Table E)	
$(C - C_A)d_A$		\$931.35
Interest rate	0.06	
Tax rate	0.01	
r	0.07	
Cr		490.00
O		414.00
M		0.00
y_A = yearly cost with 8-inch pipe		\$1835.35

For the 10-inch pipe:

C	9200	
C_A	2300	
$C - C_A$	6900	
d_A	0.1774 (opposite "5," under "6 per cent.," Table E)	
$(C - C_A)d_A$		\$1224.26
Interest rate	0.06	
Tax rate	0.01	
r	0.07	
Cr		644.00
O		146.00
M		0.00
y_A = yearly cost with 10-inch pipe		\$2014.26

Step 6. *The Decision.*—(*a*) The computation has been made on the assumption of a uniform delivery of 1 cubic foot per second during the entire five-year period, and for this assumed condition shows the 8-inch pipe to be the most economical of the three. The 10-inch pipe is second choice.

(*b*) In step 2 it is stated that there is some probability that the demand during the last two years will double. Therefore before we can advance further toward the decision we must know which is the most economical size of pipe for a delivery of 1 cubic foot per second during the first three years and 2 cubic feet per second during the last two years. The computation necessary for this information is presented in the next section, 122. Further discussion on the decision will be deferred till we reach step 6 (*b*) of §122.

122. Economic Size of Pipe-line for Varying Delivery.—This example is complete and self-contained, and at the same time it is a part of the example of §121; and it will be observed that the data below are the same as in §121 except that here we have a different assumption as to service and a correspondingly different operating cost.

Step 1. *Numerical Data.*—The period of service is five years; and during the first three years 1 cubic foot per second, and during the last two years 2 cubic feet per second of water are to be delivered.

Stock sizes of pipe	6 inches	8 inches	10 inches
First cost of 1 mile of pipe laid .. .	$5000	$7000	$9200
Salvage value at age 5 years, (25 per cent.)	$1250	$1750	$2300
Interest rate on amortization fund, per cent.	6	6	6
Interest rate on capital, per cent.........	6	6	6
Tax rate, per cent.	1	1	1
Yearly cost of operation, first 3 years. ...	$1740	$414	$146
Yearly cost of operation, last 2 years..	$6400	$1575	$531
Yearly cost of maintenance....	0	0	0

Step 2. *Outstanding Data.*—Considering this as an independent example, there is no outstanding datum. Considering this a part of the example of §121, there is the one datum stated in step 2 of §121.

Step 3. *Basis of Comparison.*—Yearly cost of service, basis A of Table VII, is chosen.

Step 4. *Formula.*—The formula proper for basis A and a period of service less than the life of the structure, is A2 of Table VII.

Step 5. *Computation.*—Since the yearly cost of operation (step 1) is not uniform over the five years, we must first reduce the irregular yearly cost to an equivalent uniform yearly cost. For this we might use Eq. 56, but since the given yearly cost is uniform during the first three years and uniform also, though

different, during the last two years, it is possible and a little quicker to use sinking-fund formulas as was done in the second solution of the example of §81. The formula there used for equivalent uniform yearly cost of operation was

$$O = a_5[O_1 w_3 + O_4(w_5 - w_3)],$$

where O_1 = given yearly cost of operation during the first three years;

O_4 = given yearly cost of operation during the last two years;

w_5 = present worth of \$1 per year for five years;

w_3 = present worth of \$1 per year for three years;

a_5 = annuity produced by \$1 principal, for five years.

Now $w_5 = 4.212$ (opposite "5," under "6 per cent.," Table D)

$w_3 = 2.673$ (opposite "3," under "6 per cent.," Table D)

$w_5 - w_3 = 1.539$

$a_5 = 0.2374$ (opposite "5," under "6 per cent.," Table F).

For the 6-inch pipe, the equivalent uniform yearly cost of operation is therefore

$$O = 0.2374(1740 \times 2.673 + 6400 \times 1.539) = 3442.45.$$

For the 8-inch pipe, the equivalent uniform yearly cost of operation is

$$O = 0.2374(414 \times 2.673 + 1575 \times 1.539) = 838.15.$$

For the 10-inch pipe, the equivalent uniform yearly cost of operation is

$$O = 0.2374(146 \times 2.673 + 531 \times 1.539) = 286.65.$$

These equivalent uniform yearly costs of operation are used for O in formula A2 (Table VII) which is

$$y_A = (C - C_A)d_A + Cr + O + M.$$

For the 6-inch pipe:

C..........	5000
C_A..........	1250
$C - C_A$.....	3750
d_A..........	0.1774 (opposite "5," under "6 per cent.," Table E)

$(C - C_A)d_A$		$665.25	
Interest rate	0.06		
Tax rate	0.01		
r	0.07		
Cr		350.00	
O		3442.45	
M		0.00	
y_A = yearly cost with 6-inch pipe			$4457.70

For the 8-inch pipe:

C	7000		
C_A	1750		
$C - C_A$	5250		
d_A	0.1774 (opposite "5," under "6 per cent.," Table E)		
$(C - C_A)d_A$		$931.35	
Interest rate	0 06		
Tax rate	0 01		
r	0.07		
Cr		490.00	
O		838.15	
M		0.00	
y_A = yearly cost with 8-inch pipe			$2259.50

For the 10-inch pipe:

C	9200		
C_A	2300		
$C - C_A$	6900		
d_A	0.1774 (opposite "5," under "6 per cent.," Table E)		
$(C - C_A)d_A$		$1224.26	
Interest rate	0.06		
Tax rate	0.01		
r	0.07		
Cr		644.00	
O		286.65	
M		0.00	
y_A = yearly cost with 10-inch pipe			$2154.91

Step 6. *The Decision.*—(*a*) If we consider the foregoing as an example distinct from the example of §121, we must conclude that the 10-inch pipe is the most economical and the 6-inch pipe is the least economical of the three, since the former has the lowest and the latter the highest yearly cost as computed above, and there are no outstanding data.

(*b*) Now to resume the discussion left incomplete in §121, step 6, (*a*). From §121, step 5, we have:

For the delivery of 1 cubic foot per second for five years,

Yearly cost with the 10-inch pipe =	$2014.26
Yearly cost with the 8-inch pipe =	1835.35
Yearly difference in favor of the 8-inch pipe =	$178.91

From §122, step 5 (above), we have:

For the delivery of 1 cubic foot per second for three years, and 2 cubic feet per second for two years,

Yearly cost with the 8-inch pipe =	$2259.50
Yearly cost with the 10-inch pipe =	2154.91
Yearly difference in favor of the 10-inch pipe =	$104.59.

Thus it appears that if we put in the 8-inch pipe and the demand double after three years, the yearly loss will be $104.59 as compared with the use of the 10-inch pipe; and that if we put in the 10-inch pipe in anticipation of a doubled demand after three years and the demand remain constant, the yearly loss will be $178.91. The *degree* of probability that there will be the doubled demand after three years is not stated (in §121, step 2); but in order that we may have complete data for rendering a decision let us assume that the chances for and against the doubled demand are even. With this assumption the choice must fall on the 8-inch pipe.

123. Main-stream Reservoir v. Gulch Reservoir.—There were two feasible reservoir sites. One was in the main stream of the drainage area. The other site was in a tributary gulch into which water from the main stream could be led by tunnel and ditch.

It was estimated that the main-stream reservoir would need to be cleared of silt every two years, at a cost of $600 for labor and $200 for lost revenue due to interrupted service; a total cost of $800 for maintenance every two years. Similarly it was estimated that the gulch reservoir would need cleaning every fifth year at a total cost of $800 for labor and loss of revenue. These and other data are given below.

Step 1. *Numerical Data.*—The service was assumed to be continued forever.

Reservoir	Main stream	Gulch
First cost	$39,200	$50,600
Life	Infinite	Infinite
Interest rate on amortization fund, per cent	5	5
Interest rate on capital, per cent	5	5
Yearly cost of upkeep, structure idle	$100	$ 50
Yearly cost of operation	500	600
Yearly cost of maintenance every second year, due to operating	800	....
Yearly cost of maintenance every fifth year, due to operating	....	800

Step 2. *Outstanding Data.*—In the case of the dam proposed for the main stream there was a possibility of a maximum flood greater than that assumed for the design of spillway, and the consequent destruction of the dam.

Step 3. *Basis of Comparison.*—The comparison was made on the basis of capitalized cost, basis B of Table VII.

Step 4. *Formula.*—With basis B and a perpetual service, the required formula is B4 of Table VII.

Step 5. *Computation.*—Before substituting in formula B4, the equivalent uniform yearly cost of maintenance was found for each reservoir in the following manner:

Let M_2 = cost of maintenance every second year, for main stream;
M_5 = cost of maintenance every fifth year, for gulch;
M = equivalent uniform yearly cost of maintenance;
d_2 = end-of-year deposit in sinking fund to redeem $1 at the end of two years;
d_5 = end-of-year deposit in sinking fund to redeem $1 at the end of five years.

Now $M_2 = 800$,
$M_5 = 800$,
$d_2 = 0.4878$ (opposite "2," under "5 per cent.," Table E),
and $d_5 = 0.1810$ (opposite "5," under "5 per cent.," Table E).

Therefore for the main-stream reservoir the equivalent uniform yearly cost of maintenance is

$$M = M_2d_2 = 800(0.4878) = \$390.24;$$

and for the gulch reservoir the equivalent uniform yearly cost of maintenance is

$$M = M_5 d_5 = 800(0.1810) = \$144.80.$$

These values of M are used below in formula B4, of Table VII which is

$$Y_\infty = y_\infty/(s - 1)$$

where

$$y_\infty = (C - C_L)d_L + Cr + O + M. \qquad \text{(A4)}$$

Since the life of each of the proposed structures is perpetual, $L = \infty$ and d_L = zero, and therefore $(C - C_L)d_L$ = zero. Therefore, for each reservoir, the capitalized cost of service is

$$Y_\infty = (Cr + O + M)/(s - 1).$$

For the main-stream reservoir:

C	39,200	
Interest rate. .	0.05	
Yearly interest.	1,960	
Upkeep, idle. . .	100	
Cr (see §78, example)		$2060
O		500
M		390.24
y_∞ = yearly cost		$2950.24
$s - 1 = (1 + 0.05) - 1 = 0.05$		
$Y_\infty = y_\infty/(s - 1)$ = capitalized cost of service with the main-stream reservoir		$59,004.80.

For the gulch reservoir:

C	50,600	
Interest rate. . .	0.05	
Yearly interest.	2,530	
Upkeep, idle. . .	50	
Cr (see §78, example)		$2580
O		600
M		144.80
y_∞ = yearly cost		$3324.80
$s - 1 = (1 + 0.05) - 1 = 0.05$		
$Y_\infty = y_\infty/(s - 1)$ = capitalized cost of service with the gulch reservoir		66,496.00
Difference in capitalized cost of service		$7,491.20

Step 6. *The Decision.*—The capitalized cost for the main-stream reservoir was found, in step 5, to be $7491.20 less than that for the gulch; and this advantage of the main-stream reservoir was weighed with the risk of destruction to the dam of this reservoir due to possibly inadequate spillway. It was considered that it was worth more than $7491.20 to avoid the risk, and accordingly the gulch reservoir was adopted.

124. Short All-tunnel Aqueduct v. Long Aqueduct of Tunnel, Ditch, and Flume.—It was required to carry water from a reservoir to irrigable lands. There were two practicable routes: one, 6–1/2 miles long, would require tunneling throughout; the other, 13 miles long, would require 7 miles of ditch, 1 mile of tunnel, and 5 miles of flume on a bench cut on the mountain side. These routes will be called "short line" and "long line," respectively. The selection was made in the following manner.

Step 1. *Numerical Data.*—The service was assumed to be perpetual. The long line required a flume the life of which was estimated at 10 years, but the other parts of the long line were considered to be everlasting; and because of this difference in lives, the data for the flume are given in a separate column.

Structure	Short line	Long line	
		Ditch, tunnel and bench	Flume
First cost	$275,000	$72,600	$57,500
Life	infinite	infinite	10 years
Salvage value at end of life	0	0	5,000
Interest rate on amortization fund	5 per cent.	5 per cent.	5 per cent.
Interest rate on capital	5 per cent.	5 per cent.	5 per cent.
Yearly cost of upkeep, idle	$ 400	$1,600	$2,200
Yearly cost of operation	1,000	1,000	500
Yearly cost of maintenance, due to operating	600	1,400	1,500

Step 2. *Outstanding Data.*—There were no conditions or circumstances of first cost or operation other than those recognized in the numerical data, which had a bearing on the choice of route.

Step 3. *Basis of Comparison.*—The yearly cost of service was taken as the basis of comparison. This is basis A of Table VII.

Step 4. *Formula.*—The basis of comparison, A, and the infinitely long period of service call for formula A4 of Table VII.

Step 5. *Computation.*—Formula A4 for yearly cost of service is

$$y_\infty = (C - C_L)d_L + Cr + O + M$$

and this reduces to

$$y_\infty = Cr + O + M$$

for all cases in which $L = \infty$, for then d_L is zero.

For the short line:

C	275,000		
Interest rate	0 05		
Yearly interest	13,750		
Upkeep, idle	400		
Cr (see §78, example)		$14,150.00	
O		1,000 00	
M		600 00	
y_∞ = yearly cost with short line			$15,750.00

For the long line:
(*a*) ditch, tunnel, and bench—

C	72,600		
Interest rate	0.05		
Yearly interest	3,630		
Upkeep, idle	1,600		
Cr		$5,230 00	
O		1,000.00	
M		1,400.00	
y_∞ = yearly cost for ditch, tunnel, and bench			$ 7,630.00

For the long line:
(*b*) flume—

C	57,500	
C_L	5,000	
$C - C_L$	52,500	
d_L	0.0795	(opposite "10," under "5 per cent.," Table E)

Brought forward		$7,630.00	$15,750.00
$(C - C_L)d_L$	$4,173.75		
Interest rate.... 0.05			
Yearly interest.. 2,875			
Upkeep, idle.... 2,200			
Cr.	$5,075.00		
O.	500.00		
M.	1,500.00		
y_∞ = yearly cost for flume		11,248.75	
y_∞ = yearly cost for long line			18,878.75
Difference in yearly costs in favor of short line			$3,128.75

Step 6. *The Decision.*—There was no outstanding datum (step 2). Consequently the decision was based on the computed yearly costs, and made in favor of the short, or all-tunnel, line.

125. Condensing Plant v. Non-condensing Plant.—A preliminary comparison was made of two proposed small steam plants for generating electric current. In one plant a simple non-condensing engine was assumed; in the other, a compound condensing engine.

Step 1. *Numerical Data.*—The service was assumed to continue forever.

Plant	Non-condensing	Condensing
First cost	$42,500	$48,500
Life (years)	25	20
Salvage value at end of life	0	0
Interest rate on amortization fund, per cent	4	4
Interest rate on capital, per cent	4	4
Insurance rate, per cent	½	½
Tax rate, per cent	½	½
Yearly cost of operation	$19,540	$13,075
Yearly cost of maintenance	425	500

Step 2. *Outstanding Data.*—The stationary engineer chosen to run the proposed plant was opposed to the compound condensing plant because he was not familiar with the type and foresaw trouble for himself if this plant should be adopted.

Step 3. *Basis of Comparison.*—The plants were compared on the basis of yearly cost of service—basis A of Table VII.

Step 4. *Formula.*—The proper formula for basis A and perpetual service is A4 of Table VII.

Step 5. *Computation.*—Formula A4 for yearly cost of service is

$$y_\infty = (C - C_L)d_L + Cr + O + M.$$

For the non-condensing plant:

C	42,500		
C_L	0		
$C - C_L$	42,500		
d_L	0.0240 (opposite "25," under "4 per cent.," Table E)		
$(C - C_L)d_L$		$1,020.00	
Interest rate	0.04		
Insurance rate	0.005		
Tax rate	0 005		
r	0.05		
Cr		2,125.00	
O		19,540.00	
M		425.00	
y_∞ = yearly cost with non-condensing plant			$23,110.00

For the condensing plant:

C	48,500		
C_L	0		
$C - C_L$	48,500		
d_L	0.0336 (opposite "20," under "4 per cent.," Table E)		
$(C - C_L)d_L$		$1,629.60	
Interest rate	0.04		
Insurance rate	0.005		
Tax rate	0.005		
r	0.05		
Cr		2,425.00	
O		13,075.00	
M		500.00	
y_∞ = yearly cost with condensing plant			17,629.60
Difference in yearly costs, in favor of condensing plant			$5,480.40

Step 6. *The Decision.*—The owners decided in favor of the

non-condensing plant. It appears that the outstanding datum (step 2) outweighed $5480.40 per annum.

126. Value of saving One Mile of Distance in a Pipe-line.—Of several feasible locations for a pipe-line, the unit costs, except those for right of way, were practically the same. Notwithstanding the fewer units of right of way required for the shorter locations, the total right-of-way costs on the shorter locations sometimes far exceeded those on the longer. On the other hand, other than right-of-way costs would be greater on the longer locations. To facilitate the selection of a location under such conditions, the following problem was solved.

Assuming that location A is 1 mile shorter than location B and that all unit costs, except for right of way, are the same for both locations; how much more can we afford to pay for right of way on location A in order to avoid the construction and operation of the extra mile on location B?

Step 1. *Numerical Data.*—One mile of 8-inch pipe to carry 2 cubic feet per second perpetually.

First cost of pipe laid	$5544
Life, years	50
Salvage value at the end of life	0
Interest rate on amortization fund, per cent	6
Interest rate on capital, per cent	6
Tax rate, per cent	1
Yearly cost of operation (pumping)	$1980
Yearly cost of maintenance	0

Step 2. *Outstanding Data.*—There were no outstanding data.

Step 3. *Basis of Comparison.*—Since the right of way was to be paid for by lump sum instead of annual sum, it was convenient to compare with this the present worth of the cost of service of the 1 mile of pipe-line. For this reason basis B (capitalized cost of service) of Table VII, was used.

Step 4. *Formula.*—With B as the chosen basis and a perpetual service (step 1), the proper formula is B4 of Table VII.

Step 5. *Computation.*—Formula B4, of Table VII, is

$$Y_\infty = y_\infty/(s-1)$$

where $$y_\infty = (C - C_L)d_L + Cr + O + M \qquad \text{(Eq. A4)}$$

C	5544	
C_L	0	
$C - C_L$	5544	
d_L	0.0034	(opposite "50," under "6 per cent.," Table E)

$(C - C_L)d_L$		\$18.85
Interest rate	0.06	
Tax rate	0 01	
r	0.07	
Cr		388.08
O		1980 00
M		0 00
y_∞ = yearly cost of service		\$2386.93

$s - 1 = 1.06 - 1 = 0.06$

Y_∞ = capitalized cost of service on 1 mile of pipe-line . . . \$39,782.17

Step 6. *The Decision.*—Since the capitalized cost of service of 1 mile of pipe-line is (in round numbers) \$40,000, and there are no outstanding data to consider, the conclusion was that it would be economical to choose, in any given similar case, the shorter location of which the right-of-way cost is greater than on a longer location, if the difference in right-of-way cost, in dollars, should not exceed 40,000 times the difference in the two location lengths expressed in miles.

127. Three Pavements Compared.—This is an example of the economic comparison of three proposed pavements:

No. 1, sheet asphalt.
No. 2, asphaltic macadam.
No. 3, bituminous cushion concrete.

Step 1. *Numerical Data.*—For the purpose of the comparison the period of service was assumed to continue forever.

Pavement	No. 1	No. 2	No. 3
First cost per square (100 sq. ft.)	\$25.00	\$16.00	\$13 00
Life, years	20	20	100
Salvage value per square at end of life	\$16.00	\$3.00	\$0.00
Interest rate on amortization fund, per cent	5	5	5
Interest rate on capital, per cent	5	5	5
Yearly cost of operation (omitted because the same for all three).			
Yearly cost of maintenance per square	\$0.40	\$0 20	\$0.167

Step 2. *Outstanding Data.*—There were no outstanding data.

Step 3. *Basis of Comparison.*—Yearly cost of service was taken as the basis of comparison. This is basis A of Table VII.

Step 4. *Formula.*—With an unlimited period of service and basis A, the proper formula is A4 of Table VII.

Step 5. *Computation.*—Formula A4 of Table VII is

$$y_L = (C - C_L)d_L + Cr + O + M.$$

For pavement No. 1:

C	25.00		
C_L	16 00		
$C - C_L$	9 00		
d_L	0.0302 (opposite "20," under "5 per cent.," Table E)		
$(C - C_L)d_L$		$0.2718	
r	0 05		
Cr		1.2500	
O (omitted)			
M		0.4000	
y_L = yearly cost per square for pavement No. 1			$1.9218

For pavement No. 2:

C	16.00		
C_L	3.00		
$C - C_L$	13.00		
d_L	0.0302 (opposite "20," under "5 per cent.,"Table E)		
$(C - C_L)d_L$		$0.3926	
r	0.05		
Cr		0.8000	
O (omitted)			
M		0.2000	
y_L = yearly cost per square for pavement No. 2			$1.3926

For pavement No. 3:

C	13.00		
C_L	0 00		
$C - C_L$	13.00		
d_L	0.0004 (opposite "100," under "5 per cent.," Table E)		
$(C - C_L)d_L$		$0.0052	
r	0.05		
Cr		0.6500	
O (omitted)			
M		0.1670	
y_L = yearly cost per square for pavement No. 3			$0.8222

Brought forward			1705.62
For automobile No. 2:			
C	2300		
C_A	1000		
$C - C_A$	1300		
d_A	0.4902 (opposite "2," under "4 per cent.," Table E)		
$(C - C_A)d_A$		$637.26	
Interest rate	0.04		
Tax rate	0.01		
r	0.05		
Cr		115 00	
O		164.00	
M		675 00	
y_A = yearly cost of automobile No. 2			$1591.26
Yearly difference in favor of No. 2			$114.36

Step 6. *The Decision.*—Since there are no outstanding data the choice is made solely on the basis of computed yearly cost; and the economic choice is therefore automobile No. 2.

Note.—In this case the required service is 6000 miles per year. Hence the cost per mile is 1705.62/6000 = $0.284+ for No. 1, and 1591.26/6000 = $0.265 + for No. 2; showing an advantage of about two cents per mile in favor of No. 2. Making the comparison thus on the basis of cost per mile is to use basis C of Table VII.

129. Steel v. Wooden Head-frame.—It was required to determine, from the following data, which of the two proposed head-frames would be the more economical for a mine.

Step 1. *Numerical Data.*—The life of the mine was estimated as 10 years; accordingly the period of service was taken as 10 years.

Head-frame	Steel	Wood
First cost	$2200	$2000
Life, years	40	10
Salvage value at the end of 10 years	0	$200
Interest rate on amortization fund, per cent	10	10
Interest rate on capital, per cent	10	10
Insurance rate, per cent	0	1
Yearly cost of operation	0	0
Yearly cost of maintenance	$5	$5

Step 2. *Outstanding Data.*—The wooden head-frame was liable to be destroyed by fire; and if it should burn, the working of the mine would be interrupted, during the rebuilding, with more or less loss of profit.

Step 3. *Basis of Comparison.*—The two structures were compared on the basis of capitalized cost of service, for reasons which will appear in step 6. This is basis B of Table VII.

Step 4. *Formula.*—Using basis B and observing that the period of service is less than the life of the steel structure and equal to the life of the wooden structure, Formula B2 (Table VII) was selected for the steel structure, and Formula B3 (Table VII)was selected for the wooden structure.

Step 5. *Computation.*—Formula B2 (Table VII) is

$$Y_A = y_A w_A$$

$$Y_A = [(C - C_A)d_A + Cr + O + M]w_A \qquad \text{(Eq. A2, Table VII).}$$

For the steel head-frame:

$$Y_{10} = [(C - C_{10})d_{10} + Cr + O + M]w_{10}.$$

C	2200		
C_{10}	0		
$C - C_{10}$	2200		
d_{10}	0.0627	(10 per cent. is beyond the range of Table E; hence, use Eq. 33, Table III: $d_{10} = (s - 1)/(s^{10} - 1)$.)	
$(C - C_{10})d_{10}$			\$137.94
Interest rate	0.10		
Insurance rate	0		
r	0.10		
Cr			220.00
O			0 00
M			5 00
y_{10} = yearly cost of service			\$362.94
w_{10}	6.146	(10 per cent. is beyond the range of Table D; hence, use Eq. 32, Table III: $w_{10} = (s^{10} - 1)/[(s - 1)s^{10}]$.)	

Y_{10} = capitalized cost of service with steel head-frame. . \$2230.63.

Formula B3 (Table VII) is $Y_n = y_n w_n$, or (Eq. A3),

$$Y_n = [(C - C_L)d_L + Cr + O + M]w_n.$$

Brought forward			2230 63

For the wooden head-frame:

$$Y_{10} = [(C - C_{10})d_{10} + Cr + O + M]w_{10}.$$

C	2000		
C_{10}	200		
$C - C_{10}$	1800		
d_{10}	0.0627 (see d_{10}, above)		
$(C - C_{10})d_{10}$		$112.86	
Interest rate	0.10		
Insurance rate	0.01		
r	0.11		
Cr		220.00	
O		0.00	
M		5.00	
y_{10} = yearly cost of service		$337.86	
w_{10}	6.146 (see w_{10}, above)		
Y_{10} = capitalized cost of service with wooden head-frame			2076.48
Difference in capitalized costs			$154.14

Step 6. *The Decision.*—The capitalized cost of the wooden head-frame is (step 5) $154.14 less than that of the steel head-frame; but it was considered to be worth more than this sum to avoid the risk of interruption from fire (step 2), and the steel head-frame was chosen. (Note that the loss on the head-frame itself would be covered by insurance.)

130. Fifty-five-ton v. Seventy-ton Steam-shovel.—Which is the more economical for steady work in a phosphate mine, the 55-ton or the 70-ton steam-shovel?

Step 1. *Numerical Data.*—The period of service is taken as 20 years, which is the life of the lighter shovel.

Steam-shovel	55-ton	70-ton
First cost	$7,000	$10,000
Life, years	20	25
Salvage value at the age of 20 years.	$500	$2,000
Interest rate on amortization fund, per cent	6	6
Interest rate on capital, per cent	6	6
Tax rate, per cent	1	1
Insurance rate, per cent.	½	½
Minimum upkeep, per cent	½	½
Yearly cost of operation	$8,700	$10,500
Yearly cost of maintenance	$800	$1,000
Yearly output (cubic yards)	130,000	180,000

Step 2. *Outstanding Data.*—There are no outstanding data.

Step 3. *Basis of Comparison.*—Since the outputs of the two shovels will not be equal, they must be compared on the basis of unit cost of service (basis C of Table VII) or on the basis of capitalized cost per unit of service per year (basis D of Table VII). Basis C involves less computation than basis D, and is, moreover, a measure of cost in common use.

Step 4. *Formula.*—The period of service (step 1) is equal to the life of the 55-ton shovel, and less than the life of the other; hence, with basis C (step 3), C3 (Table VII) is the formula to be used for the former and C2 (Table VII) is that to be used for the latter.

Step 5. *Computation.*—Formula C3 (Table VII) is $v_n = y_n/U$, or, from Eqs. C3 and A3,

$$v_n = [(C - C_L)d_L + Cr + O + M]/U.$$

For the 55-ton shovel, this equation becomes

$$v_{20} = [(C - C_{20})d_{20} + Cr + O + M]/U$$

C	7000		
C_{20}	500		
$C - C_{20}$	6500		
d_{20}	0.0272	(opposite "20," under "6 per cent.," Table E)	
$(C - C_{20})d_{20}$			$176.80
Interest rate	0 06		
Tax rate	0.01		
Insurance rate	0.005		
Minimum upkeep	0.005		
r	0.08		
Cr			560.00
O			8700.00
M			800.00
y_{20} = yearly cost of service			$10,236.80

U = 130,000

v_{20} = cost per unit of service with the 55-ton shovel
$= y_{20}/U$ = $0.07874, or.................... 7.874 cents.

Formula C2 (Table VII) is $v_A = y_A/U$, or, from Eqs. C2 and A2,

$$v_A = [(C - C_A)d_A + Cr + O + M]/U.$$

For the 70-ton shovel, this equation becomes

$$v_{20} = [(C - C_{20})d_{20} + Cr + O + M]/U.$$

Brought forward			7.874 cents.
C	10,000		
C_{20}	2,000		
$C - C_{20}$	8,000		
d_{20}	0.0272 (opposite "20," under "6 per cent.," Table E)		
$(C - C_{20})d_{20}$		$217.60	
Interest rate	0 06		
Tax rate	0.01		
Insurance rate	0.005		
Minimum upkeep	0.005		
r	0.08		
Cr		800.00	
O		10,500.00	
M		1,000.00	
y_{20} = yearly cost of service		$12,517.60	
U = 180,000			
v_{20} = cost per unit of service with the 70-ton shovel = y_{20}/U = $0.06954, or			6.954 cents.
Difference in unit costs			0 920 cents.

Step 6. *The Decision.*—As computed above, the larger shovel will do the work for about 9/10 cent less than the smaller shovel will, and, since there are no outstanding data (step 2), the decision must be in favor of the 70-ton shovel.

131. Diesel v. Non-condensing Engine.—The question here was whether to renew a non-condensing steam engine generating set of 250 kv-a. capacity or replace it by a Diesel engine generating set of the same capacity.

Step 1. *Numerical Data.*—The period of service was taken as 15 years, since that was the estimated life of each of the engine sets.

Engine set	Diesel	Non-condensing
First cost	$28,000	$23,366
Life, years	15	15
Salvage value at the end of life	$1,000	$1,000
Interest rate on amortization funds, per cent.	5	5
Interest rate on capital, per cent	5	5
Yearly cost of operation and maintenance (with an output of 900,000 kw-hr. per annum).	$7,780	$15,700

Step 2. *Outstanding Data.*—As the contemplated Diesel engine would be made to order in Germany, the replacement would take four months longer than the renewal.

Step 3. *Basis of Comparison.*—The comparison was made on the basis of yearly cost of service—basis A of Table VII.

Step 4. *Formula.*—Since the chosen basis of comparison is A and the period of service is taken equal to the life of each structure, the proper formula is A3 of Table VII.

Step 5. *Computation.*—Formula A3 of Table VII is

$$y_n = (C - C_L)d_L + Cr + O + M.$$

For the Diesel engine set:

C	28,000		
C_L	1,000		
$C - C_L$	27,000		
d_L	0 0463	(opposite "15" under "5 per cent.," Table E)	
$(C - C_L)d_L$			$1250 10
r	0 05		
Cr			1400 00
$O + M$			7780 00
y_n = yearly cost of service with Diesel engine			$10,430.10

For the non-condensing steam engine set:

C	23,366		
C_L	1,000		
$C - C_L$	22,366		
d_L	0.0463	(opposite "15," under "5 per cent.," Table E)	
$(C - C_L)d_L$		$1,035.55	
r	0.05		
Cr		1,168.30	
$O + M$		15,700.00	
y_n = yearly cost of service with the non-condensing engine			17,903.85
Difference in yearly cost of service in favor of the Diesel engine set			$7,473.75

Step 6. *The Decision.*—The computed advantage of the Diesel engine set was considered as more than offsetting the disadvantage of delay in obtaining the same, and the Diesel engine set was ordered.

132. Economic Diameter of Tunnel for Carrying Water to Hydro-electric Plant.—The problem is here solved by the method of trial. The yearly cost of tunnel service is computed for one

diameter after another, until the results show that the economic diameter has been passed.

Step 1. *Numerical Data.*—The length of the proposed tunnel is 50,000 feet; the quantity of water flowing per second is 800 cubic feet. The period of service is assumed to be perpetual, and the interest rate 7 per cent.

	Tunnel data				
Diameters	9 5 ft.	10 ft.	10 5 ft.	11 ft.	11.5 ft.
First cost	$2,425,500	$2,557,500	$2,687,500	$2,824,000	$2,975,500
Life	infinite	infinite	infinite	infinite	infinite
Yearly cost of maintenance.	0	0	0	0	0
Yearly cost of operation.[1]	$72,000	$55,600	$43,800	$34,800	$28,000

Step 2. *Outstanding Data.*—Some losses pertaining to operation are necessarily or more satisfactorily given consideration as outstanding data (§84); but in this example the only operating loss mentioned—that due to tunnel friction—is given for each of the listed diameters, and so can be, and is, recognized in the computation for cost of service (step 5) instead of being relegated for consideration as an outstanding datum.

Step 3. *Basis of Comparison.*—The yearly cost of service is taken as the basis of comparison. This is basis A of Table VII.

Step 4. *Formula.*—The life of the tunnel is infinite in each case, and the period of service is assumed to be perpetual; therefore the proper formula for computing cost of service is, in each case, A4a of Table VII.

Step 5. *The Computation.*—Formula A4a of Table VII is

$$y_\infty = Cr + O + M.$$

Diameters	9 5 ft.	10 ft.	10.5 ft.	11 ft.	11.5 ft.
C	2,425,500	2,557,500	2,687,500	2,824,000	2,975,500
r	0 07	0 07	0.07	0.07	0.07
Cr	169,785	179,025	188,125	197,680	208,285
O	72,000	55,600	43,800	34,800	28,000
M	0	0	0	0	0
y_∞	$241,785	$234,625	$231,925	$232,480	$236,285

Step 6. *The Decision.*—The computation shows that y_∞, the

[1] No payments are to be made on account of operation, but there will be an annual loss of power, due to tunnel friction, and the estimated money value of this annual power loss is here entered as a yearly cost of operation.

yearly cost of service, decreases as the diameter of the tunnel increases up to 10.5 feet; beyond that diameter the yearly cost increases as the diameter increases. As there are no outstanding data (the loss due to friction having been entered in the formula as a cost of operation) the diameter of 10.5 feet is the most advantageous. (Practically the same result could be obtained by computing yearly costs for diameters of 9, 10, 11, and 12 feet, and plotting a curve with the diameters as abscissas and the corresponding yearly costs as ordinates. The ordinate of the lowest point on the curve would be very nearly 10.5 feet.)

133. Diesel Engine v. Steam Turbine.—The following is a comparison made between a Diesel engine and a steam turbine for a 10,000-kw. power station.

Step 1. *Numerical Data.*—The period of service is not given.

	Diesel	Turbine
First cost of station per kw. capacity	\$70	\$50
Fixed charges (see schedule of §72) in per cent. of first cost	13	13
Rated hourly capacity, kw.	10,000	10,000
Load factor, per cent.	60	60
Production cost of one kw-hr. (operation and maintenance), cents	0 86	0 90

Step 2. *Outstanding Data.*—No outstanding data.

Step 3. *Basis of Comparison.*—Comparison is made on the basis of yearly cost of service. This is basis A of Table VII.

Step 4. *Formula.*—With basis A, we should use one of the formulas of line A, Table VII. We are not given the estimated life of either structure (step 1), nor do we know the period of service; hence, apparently, there is no way of determining which formula of line A (Table VII) to use for the computation of yearly cost of service. But, looking at all the formulas on line A, we see that they vary only in the first two terms of the right-hand member—in the two terms which together give the yearly fixed charges (see schedule of §72). So, since the yearly fixed charges are given (in per cent. of first cost, at least) in step 1, we can use any one of the formulas of line A. Let us use formula A3.

Step 5. *Computation.*—Formula A3 of Table VII is

$$y_n = (C - C_L)d_L + Cr + O + M.$$

For the Diesel engine:

$(C-C_L)d_L+Cr=13$ per cent. of $70\times 10{,}000=\$\ 91{,}000$
$O+M=0.0086\ (10{,}000\times 0.60\times 24\times 365)\ =\$452{,}016$

$y_n=$ yearly cost with Diesel engine. . . . . $543,016.

For the steam turbine:

$(C-C_L)d_L+Cr=13$ per cent. of $50\times 10{,}000=\ 65{,}000$
$O+M=0.0090(10{,}000\times 0.60\times 24\times 365)\ =\ 473{,}040$

$y_n=$ yearly cost with steam turbine. 538,040

Difference in yearly cost in favor of turbine. . . . . . . $ 4,976

Step 6. *The Decision.*—Since there are no outstanding data (step 2), the choice must fall on the steam turbine, the yearly cost of which is $4976 less than that of the Diesel engine.

134. Condensing v. Non-condensing Steam Equipment.—The question here is: will the first cost of a condenser be more than offset by the reduction which the condenser will effect in the running expenses of the steam plant?

Step 1. *Numerical Data.*—The period of service is not stated.

First cost of condenser equipment.	$4000
Fixed charges (see schedule of §72) in per cent. of first cost, per cent	15
Output of station = 1000 h.p. 10 hr. per day for 300 days of the year.	
Oil consumption without condenser, lb. per h.p.	1.3
Oil consumption with condenser, lb. per h p.	1.1
Oil costs 90 cents per bbl. of 336 lb.	

Step 2. *Outstanding Data.*—There are no outstanding data.

Step 3. *Basis of Comparison.*—The yearly cost of condenser service is compared with the yearly saving effected by the use of the condenser.

Step 4. *Formula.*—Since yearly fixed charges are given (in per cent. of first cost), any one of the formulas of line A, Table VII, will serve for computing the yearly cost of condenser service. We will use formula A3.

Step 5. *Computation.*—Formula A3 of Table VII is

$$y_n = (C - C_L)d_L + Cr + O + M.$$

$(C - C_L)d_L + Cr = 15$ per cent. of $4000 = \$600$
$O + M\ =\ 0$

y_n = yearly cost of service of condenser $600

Yearly saving in fuel oil due to use of condenser is

$$\frac{(1.3 - 1.1)(1000 \times 10 \times 300)}{336}0.90 = \qquad 1602$$

Difference, in favor of condenser. $1002

Step 6. *The Decision.*—From the results above, it appears that there would be a net saving of $1002 per year from the use of the condenser. Therefore, since there are no outstanding data, the condenser should be provided.

PART. IV

BIBLIOGRAPHY AND DEPRECIATION AND LIFE TABLES

APPENDIX A

SELECT LIST OF PAPERS ON HIGHER PROBLEMS OF ECONOMIC SELECTION

For the treatment of problems of economic selection more advanced than those presented in the present work, the reader is referred to the books and articles listed below.

Electric Lighting Conductors. *Electrical Review* (London), Feb. 28, 1885, p. 189.

Kelvin's Law of Economy in Conductors. Frederic A. C. Perrine. Chapter VIII of his Conductors for Electrical Distribution, 1903, D. Van Nostrand Company, New York.

The Maximum Distance to which Power can be Economically Transmitted. Ralph D. Mershon. *American Institute of Electrical Engineers*, 1904, vol. 23, p. 759.

Commercial Economy in Steam and Other Thermal Power-Plants. Robert H. Smith. London, A. Constable & Co., Ltd., 1905.

Economical Sizes for Cast-iron Force Mains. W. L. Butcher. *Engineering Record*, May 13, 1905, p. 558.

A Solution of the Problem of Determining the Economic Size of Pipe for High-pressure Water-power Installation. Arthur L. Adams. *Transactions American Society of Civil Engineers*, Dec., 1907, vol. 59, pp. 173–194.

Pipe Lines for Hydraulic Power Plants. Arthur Jobson. *Engineering Record*, Dec. 21, 1907, p. 673.

To Determine when Repairs Have Grown so Great as to Justify Renewal. Halbert P. Gillette. Pages 27–33 of his Handbook of Cost Data, 1910, The Myron C. Clark Publishing Co., Chicago and New York.

Deduction of a New Rational Formula for the Most Economic

Length of Each of a Series of Bridge Spans. Halbert P. Gillette. *Engineering-Contracting*, Jan. 4, 1911, p. 21.

Deduction of a General Formula for Determining the Most Economic Size of Pipe to Carry Pumped Water. Halbert P. Gillette. *Engineering-Contracting*, Jan. 25, 1911, p. 113.

A New Method for Determining Economical Pipe Diameters. Frank H. Carter. *Engineering Record*, Dec. 16, 1911, p. 722.

A Study of Economic Conduit Location. C. E. Hickok. *Transactions American Society of Civil Engineers*, vol. 77, Dec., 1914, pp. 778–787.

Comparative Value of Cross-ties of Different Materials. Walter Loring Webb. In his Economics of Railroad Construction, second edition, p. 175. 1912, John Wiley & Sons, New York. Also S. Whinnery in *Railroad Gazette*, Nov. 11, 1904, p. 537.

Economic Limit of Pavement Repairs. George H. Norton. *The Cornell Civil Engineer*, January, 1915, pp. 123–130.

APPENDIX B

SELECT LIST OF TREATISES AND TABLES

The chapter heads of each treatise listed below are given for the double purpose of acquainting the student with the variety of the subjects which are important to engineers, but of which he does not ordinarily know the existence; and of aiding the engineer to determine in advance whether any given one of the treatises will meet his needs. For like reasons each table of each of the table books listed is fully described.

1. TREATISES

Funds and Their Uses.—Frederick A. Cleveland, New York, D. Appleton and Company, 1904, 304 pages, $1.25.

Introductory.—Part I: Money funds—credit funds—instruments of transfer of credit funds. Part II: Funds obtained by gift and inheritance—funds obtained by exchange—funds obtained by sales of commercial credit—funds obtained by sales of long-time paper. Part III: The U. S. Treasury—the savings-bank—the building loan association—the commercial bank—the trust company—the broker and the brokers' board—the insurance company.

Illustrated by more than 100 cuts showing various financial instruments such as notes, checks, drafts, stock certificates, bonds, etc., etc.

Investment Bonds.—Frederick Lownhaupt, New York, G. P. Putnam's Sons, 1908, 253 pages, $1.75.

Introductory—classification—denominations—processes of issue and negotiation—limitations—legality—market—holdings—interest—security—guarantee—maturity—mortgage—lease—taxation—refunding—exchange—bonds as collateral—default and repudiation—reorganization—voting power—municipal bonds—sinking fund—serial bonds—balance sheet.

The Mathematical Theory of Investment.—Ernest Brown Skinner, Boston, Ginn and Company, 1913, 245 pages, $2.25.

Part I: Progressions—limits and series—logarithms—graphical representation. Part II: Interest—annuities—amortization—valuation of bonds—sinking funds and depreciation—building and loan associations. Part III: The theory of probability—

life annuities—some problems in life insurance. Part IV: Tables (see under "Tables," below).

Public Utilities, Their Cost New and Depreciation.—Hammond V. Hayes, New York, D. Van Nostrand Company, 1913, 262 pages, $2.

Property valuations: general considerations—replacement costs of physical property—determination of replacement cost—value as going concern—values of good will and franchises—original cost—commercial value—the worth of service to the consumer—reserves for depreciation—life of plant—depreciation—fair present value: rates—fair present value: condemnation or sale—general considerations relative to the regulation of public utility undertakings.

Valuation of Public Utility Properties.—Henry Floy, New York, McGraw-Hill Book Company, Inc., 1912, 390 pages, $5.

Introduction—glossary—public service commissions—making an appraisal—structural costs—development expenses, intangible expenses, non-physical costs, overhead expenses—franchises, good will, going value, contracts—depreciation—appraisals of public utility properties in Greater New York—examples of important appraisals.

Engineering Valuation of Public Utilities and Factories.—Horatio A. Foster, New York, D. Van Nostrand Company, second edition, 1913, 345 pages, $3.

Introduction—value—purposes of valuation—directions for the valuation of tangible property—instructions for valuation—forms for use in making a valuation—the cost of valuing a property—value of good will, going concern, or going value—depreciation—amortization—handling of depreciation funds—appreciation—franchise—capitalization—control of public utilities—court decisions—bibliography.

Valuation of Public Service Corporations.—(Legal and economic phases of valuation for rate making and public purchase)—Robert H. Whitten, New York, The Banks Law Publishing Co., 1912, 797 pages, $5.50.

Purpose of valuation—fair value for rate purposes—market value as a standard for rate purposes—cost of reproduction as a standard of value for rate purposes—actual cost as a standard of value for rate purposes—valuation of land—pavement over mains—property donated or acquired without cost—property constructed out of surplus—unused property—average price

v. present price—overhead charges—discount on bonds—working capital—piecemeal construction—adaptation and solidification—physical depreciation—cost-new *v.* cost-less-depreciation—functional depreciation—annual depreciation allowance—going concern in purchase cases—going concern in rate cases—going concern as the value of a created income—going value rule of Wisconsin Railroad Commission—the theory of going concern value—franchise value in purchase cases—franchise value in rate cases—appraisal of franchise value—the theory of franchise value—rate of return—rules for appraisers in Maine condemnation cases—bibliography of valuation and depreciation.

Valuation of Public Service Corporations.—(Legal and economic phases of valuation for rate making and public purchase: Supplement.) Robert H. Whitten, New York, The Banks Law Publishing Co., 1914, pp. 799–1443, $5.50.

Material is arranged under the same chapter headings as in the preceding volume, with the addition of: unit prices—contract, water power, and patent rights—tables of cases cited—index (to both volumes).

The Law of Operations Preliminary to Construction in Engineering and Architecture.—John Cassan Wait, New York, John Wiley & Sons, 1905, 712 pages, cloth $5, sheep $5.50.

Part I: Introduction; property defined—ownership of lands; estates—title to property; how acquired—conveyances of land; essential elements of deeds. Part II: Water; riparian owner; appropriation of water—waters for irrigation in arid countries—detention of water of streams; mills and mill rights—diversion and obstruction of waters; streams—protection of banks and structures from waters—supply of water and ice; water companies and waterworks—water; rights in regard to surface-waters—fouling and pollution of surface-waters and streams—navigable waters; public and private rights in navigable waters—subterranean or underground waters—oil and gas; ownership and appropriation of oil and gas—electricity; property rights affected by the use and discharge of electricity—light and air incident to land—property rights defined by boundary lines; lateral support—interference or invasion of property rights by surveyors. Part III: Boundaries in general; how described, established, and maintained—boundaries on waters; shifting character; accretion, erosion, reliction, and reclamation—boundaries on waters; land bounded by, along, upon, or on a stream or the bed, bank, beach

or shore—boundaries on lakes and ponds—boundaries of islands—boundaries on streets and roads—boundaries determined by arbitration—boundaries established by agreement or acquiescence—adverse possession; title and boundaries to land affected by it—construction, interpretation, and application of descriptions—description; conflict of calls—determination and proof of boundaries. Part IV: Easements in general—license, revocable and irrevocable—prescription and prescriptive rights—dedication of rights in land to public—easements; rights of way in general—right of way of railroad—rights of way of street railway—right of way for telegraph and telephone lines—rights of way in conduits, pipe-lines, etc., for water, oil, air, gas, and electricity. Part V: Character and kinds of franchises.

Engineering and Architectural Jurisprudence.—John Cassan Wait, New York, John Wiley & Sons, 1907, 985 pages, cloth, $6, sheep $6.50.

Part I (Law of contracts in general): Essential elements of a contract; legal and illegal contracts; the parties to a contract—law of contracts; essential elements of a contract; the consideration—mutual consent or mutual assent—general statutes limiting the law of contracts. Part II (Bids and bidders): Rights and liabilities of bidders for public work—bids and bidders; work for private parties. Part III (A construction contract. Its phraseology, terms, conditions, stipulations, provisions, and requirements, and their interpretation, construction, and force): Introduction; authority to contract; requirements—plans and specifications—the ownership, disposal, inspection, acceptance or rejection of materials of construction—commencement and completion of work—the Engineer or Architect an arbitrator, umpire, or referee—Engineer or Architect as a quasi-arbitrator, umpire, or referee—recovery by contractor without producing Engineer's certificate—certificate and estimate of Engineer or Architect conclusive on both parties to the contract—Engineer's or Architect's certificate—delegation of Engineer's or Architect's duties to assistants—interest of the Engineer or Architect in common with the owner or contractor—matters of doubt and dispute submitted to arbitration—extra work or extras, alterations, additions, omissions, and substitutions—custom and usage in construction work—owner's liability for acts of contractor—non-performance of contract; impossibility of performance—non-performance of contract; breach or rescision—payment;

progress and final payments; preliminaries to payments. Part IV (Engineer's and Architect's employment): Employment or engagement of Engineer or Architect—property of Engineers or Architects in designs and inventions—liability of Engineer or Architect as a professional man—liability of Engineer or Architect when his functions are judicial or discretionary—liability of an Engineer or Architect when a public officer—compensation of Engineers and Architects—employment of an Engineer or Architect as an expert witness.

Financing an Enterprise.—Francis Cooper, New York, The Ronald Press, 1907, second edition, 543 pages, 2 volumes, $4.

Part I (The enterprise): Introductory—conditions of financing—methods of financing—requisites of a successful enterprise—relative status of the undertaking, capital, and management. Part II (Investigation of an enterprise): Importance of investigation—methods and results of investigation—speculative and non-speculative enterprises—investigation of a non-speculative enterprise—investigation of a speculative enterprise—experimental work and model-making. Part III (Protection of an enterprise): Patents—trade-marks, trade-names, and copyrights—secret processes—monopolies. Part IV (Capitalization of an enterprise): Capitalization; legal status; functions—value as a basis of capitalization—capitalization not based on value—capitalization based on present values; new enterprises—capitalization based on present values; going concerns; good will—capitalization based on profit probabilities—capitalization based on profit possibilities—excessive capitalization. Part V (Presentation of an enterprise): Manner and matter of presentation—preparation for presentation—the prospectus and other papers—the presentation; private or public—private presentation among friends—private presentation among strangers—public presentation by circular letters—public presentation by newspaper and magazine advertising. Part VI (Special features of promotion): Trust fund guarantees—guaranteed stock, bonds and dividends—underwriting—"promoters" and "financiers"—commissions and bonuses—legal assistance—adaptation of corporate features. Appendix: The corporate system—the investor's questions.

Bibliography on Valuation of Public Utilities.—*Transactions American Society of Civil Engineers*, vol. 76, December, 1913. Reprinted in *Bulletin* No. 159 (September, 1913) of American Railway Engineering Association.

2. TABLES

The Mathematical Theory of Investment.—In this book, which has been described on p. 151, are the following tables:

I. Number of each day in the year counting from January 1.

II. Exact interest at 5 per cent. for times from 1 to 365 days on amounts $1000 to $9000 by $1000 (one page); to 4 decimal places.

III. Compound amount on 1. 1 to 100 years by years; rates 1–1/4, 1–1/2, 2, 2–1/2, 3, 3–1/2, 4, 4–1/2, 5, and 6 per cent.; to 7 decimal places.

IV. Present value of 1. Same range as for III.

V. Amount of 1 per annum. Same range as for III.

VI. Present value of 1 per annum. Same range as for III.

VII. The annuity that 1 will purchase. Same range as for III.

VIII. Compound amounts for times less than one year. 1/12, 1/4, and 1/2 years; 1–1/4, 1–1/2, 2, 2–1/2, 3, 3–1/2, 4, 5, 6, and 7 per cent.; to 7 decimal places.

IX. Nominal rate corresponding to given effective rate. Compounding monthly, quarterly, and semi-annually; effective rates 1–1/4, 1–1/2, 2, 2–1/2, 3, 3–1/2, 4, 5, 6, and 7 per cent.; to 7 decimal places.

X. Ratio of effective rate to nominal rate. Same range as for IX.

XI. American experience table of mortality.

XII. Commutation columns: American experience, 3–1/2 per cent.

Engineer's Valuing Assistant.—H. D. Hoskold, London, New York, and Bombay; Longmans, Green, and Co., 1905, 185 pages, $3.

Construction and use of valuation tables, with rules and formulas; and the following tables:

I. Amount of 1 in any number of years to 100, at 1/2, 3/4, 1, 1–1/4, 1–1/2, 1–3/4, 2, 2–1/4, 2–1/2, 2–3/4, 3, 3–1/4, 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, and 25 per cent. Calculated to 10 places of decimals to 9 per cent., 6 places to 15 per cent., and 5 places to 25 per cent.

II. Amount of 1 in any number of years to 100, at the rate of 3 per cent. Half-yearly and quarterly payments. Calculated to 10 places of decimals.

III. Amount of 1 per annum in any number of years to 100, at the rates of 1/2, 3/4, 1, 1–1/4, 1–1/2, 1–3/4, 2, 2–1/4, 2–1/2, 2–3/4, 3, 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, 8, 9, and 10 per cent. Calculated to 10 places of decimals.

IV. Present value of 1 due *n* years hence, at 3, 3–1/2, 4, 4–1/2, 5, 6, 7, 8, 9, and 10 per cent. to 100 years, and at 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, and 25 per cent. to 50 years. Calculated to 8 places of decimals.

V. Redemption funds necessary to produce 1 in *n* years, up to 100, at the rates of 1–1/2, 2, 2–1/2, 3, 3–1/4, 3–1/2, 4, 4–1/4, 4–1/2, and 5 per cent., calculated to 10 places of decimals; and for rates of 10, 12, 15, 18, and 20 per cent., calculated to 10 places of decimals, and for 50 years. Also, for rates of 3, 3–1/4, 3–1/2, 3–3/4, 4, 4–1/4, 4–1/2, 4–3/4, and 5 per cent., payments being made half-yearly and quarterly; calculated to 6 places of decimals, and to 100 years.

VI. Present value of 1 per annum, redemption of capital being at 2–1/2 per cent., and interest at the rates of 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, and 25 per cent.; calculated to 8 places of decimals and to 100 years.

VII. Present value of 1 per annum, redemption of capital being at 3 per cent., and interest at the rates of 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, and 25 per cent.; calculated to 8 places of decimals and to 100 years.

VIII. Present value of 1 per annum, redemption of capital being at 3-1/2 per cent., and interest at the rates of 4, 5, 6, 8, 10, 12, 15, 18, 20, and 25 per cent.; calculated to 8 places of decimals and to 100 years.

IX. Present value of 1 per annum, redemption of capital being at 4 per cent., and interest at the rates of 5, 6, 8, 10, 12, 15, 16, 18, 20, and 25 per cent.; calculated to 8 places of decimals and to 100 years.

X. Present value of 1 per annum, deferred 1, 2, 3, and 4 years, redemption of capital being at 3 per cent., and interest at the rates of 4, 5, 6, 8, 10, 12, 15, 18, and 20 per cent.; calculated to 6 places of decimals and to 100 years.

XI. Present value of 1 per annum, deferred 1, 2, 3, and 4 years, redemption of capital being at 3–1/2 and 4 per cent., and interest

at the rate of 20 per cent.; calculated to 6 places of decimals and to 100 years.

XII. Comparison of the difference in value between the *old* or *ordinary* tables of present values, and a portion of the *new* tables calculated for this work, which allow one rate of interest on capital and another rate for its redemption, for rates of interest at 4, 5, 8, 10, 12, 15, 18, and 20 per cent., showing the amount and rate per cent. lost on the purchase of every 1 annuity, by the use of the *old* tables; calculated to 5 places of decimals and to 50 years.

XIII. Present value of 1 per annum in *n* years, redemption of capital being at 2 and 2–1/2 per cent.; with interest at the same rates per cent.; calculated to 5 places of decimals and to 100 years.

XIV. Present value of 1 per annum in *n* years; redemption of capital being at 3 per cent., with interest allowed upon capital at the rate of 30, 35, 40, and 45 per cent. per annum.

XV. Present value (or years' purchase) of 1 per annum in *n* years, after *t* years of deferrence; redemption of capital being at the rate of 3 per cent., with interest allowed to a purchaser at 30 per cent. per annum, deferred 1, 2, 3, 4, and 5 years.

XVI. Present value (or years' purchase) of 1 per annum in *n* years, after *t* years' deferrence; redemption of capital being at 3 per cent., with interest allowed to a purchaser at 35 per cent. per annum, deferred 1, 2, 3, 4, and 5 years.

XVII. Present value (or years' purchase) of 1 per annum in *n* years, after *t* years' deferrence; redemption of capital being at 3 per cent., with interest allowed to a purchaser at 40 per cent. per annum, deferred 1, 2, 3, 4, and 5 years.

XVIII. Present value (or years' purchase) of 1 per annum in *n* years, after *t* years' deferrence; redemption of capital being 3 per cent., with interest allowed to a purchaser at 45 per cent., deferred 1, 2, 3, 4, and 5 years.

Robinsonian Bond and Investment Tables.—J. Watts Robinson. Published by the Author, Brookline, Mass., 1910, third edition, $5.

Robinsonian Bond Values.—Years to maturity, 1 to 50; interest on bond and investment paid semi-annually; bond rates of 3, 3–1/2, 3.65, 4, 4–1/2, 5, 5–1/2, 6, 6–1/2, 7, 7–1/2, 8, 9, and 10 per cent.; annual investment rates of 2, 2–1/8, 2–1/4, 2–3/8, and so on by eighths to 7, also 7–1/4, 7–1/2, 7–3/4, 8, 8–1/4, 8–1/2, 9, 9–1/2, and 10 per cent. Values are computed on the

hypothesis that the reinvestments of interest necessary to compensate for the premium or discount at which a bond is bought are here invariably made at the rate of 4 per cent. per annum payable semi-annually. Values given to the nearest mill, for $100 bond. Also the following tables:

1. Amount of $1 for any time from 1 to 100 years at rates of 4/12, 5/12, 6/12, 7/12, 8/12, 9/12, 10/12, 11/12, 1, 1–1/4, 1–1/2, 1–3/4, 2, 2–1/4, 2–1/2, 2–3/4, 3, 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, 8, and 10 per cent., to 5 decimal places.

2. Amount of 1 for each year for any time from 1 to 100 years, at rates the same as in Table 1, to 5 decimal places.

3. Present worth of 1 for any time from 1 to 100 years, at rates the same as in Table 1, to 6 decimal places.

4. Present worth of 1 each year for from 1 to 100 years, at rates the same as in Table 1, to 5 decimal places.

5. Deposit to be made in sinking fund at the *beginning* of each year for any number of years from 1 to 100, to amount to 1, at rates 1–1/2, 1–3/4, 2, 2–1/4, 2–1/2, 3, 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, and 8 per cent., to 6 decimal places.

6. The end-of-year annuity for any number of years, which a principal of 1 will yield at rates of 1, 1–1/2, 2, 2–1/2, 3, 3–1/2, and 4 per cent. for from 1 to 100 years, and 4–1/2, 5, 5–1/2, 6, 7, 8, and 10 per cent., for from 1 to 50 years, to 5 decimal places.

7. Present worth of 1 for any time within a year, at rates of 2, 2–1/2, 3, 3–1/2, 4, 4–1/2, 5, 5–1/2, 6, 7, 8, 9, and 10 per cent., to 6 decimal places.

8. Deposit to be made in sinking fund at *end* of each year for from 1 to 100 years, to amount to 1, at rates of 1–1/2, 1–3/4, 2, 2–1/4, 2–1/2, 2–3/4, 3, 3–1/2, 4, 4–1/2, 5, 5–1/2, and 6 per cent., to 6 decimal places.

9. Deposit to be made in sinking fund at the *beginning* of each *half-year* for from 1 to 100 half-years, to amount to 1, at rates of 1–1/2, 1–3/4, 2, 2–1/4, 2–1/2, 2–3/4, and 3 per cent., to 6 decimal places.

10. Income derived from bonds that pay interest semi-annually.

Also a simple-interest table for 360-day year, and another for 365-day year, both occupying less than one page.

APPENDIX C

DEPRECIATION RATES AND LIFE TABLES

The following tables are inserted for the purpose of assisting those who have occasion to estimate the number of years in the lives of structures. By dividing "depreciation per cent. per year" into the difference between 100 and salvage value per cent., the corresponding life, in years, is obtained.

(NOTE.—The matter of this appendix is taken bodily from Foster's "Engineering Valuation of Public Utilities and Factories," second edition, 1913, by the kind permission of the publishers, Messrs. D. Van Nostrand Co.)

In a paper read at the American Institute of Electrical Engineers in Chicago, June, 1911, Henry Floy gave the following compilation of rates which have the added value that they have been accepted by some authority. They do not include maintenance, but cover theoretical depreciation. The list applies especially to electric railways, but many items would also be found in other industries.

Approved Rates used in Estimating Theoretical Depreciation

By Henry Floy, *Trans. Am. Inst. E. E.*, June, 1911

(Maintenance not included)

Property	Depreciation per cent per year Straight line	Authority	Remarks
Aërial lines	5	St Louis P S. C.	Union Elec L. & P. Co.
Air brakes	5	Wisconsin R. R. C.	
Air compressors	4–5	Traction Val. Comm	Chicago Con. Tract. Co.
Arc lamps	6⅔	Wisconsin R. R. C.	
	15	Arbitrators	Street Lighting Controversy, Atlanta, Ga., 1899
	8	St. Louis P. S C.	Union Elec L. & P Co.
Belting	5	Wisconsin R R C.	
Boilers	3½–4	Traction Val. Comm	Chicago Con. Tract Co.
	10	B. J. Arnold	Coney Island & Brooklyn adopted by P. S. C , N. Y.
(Water tube)	5	Henry Floy	3d Ave Case, adopted by P. S C , N. Y.
(Fire tube)	3⅓–6⅔	Wisconsin R. R. C.[1]	
(Water tube)	5	Wisconsin R. R. C.[1]	
(Fire tube)	10	Arbitrators	Street Lighting Controversy, Atlanta, Ga , 1899.
	6⅔	St. Louis P. S. C.	Union Elec. L. & P. Co.

[1] For electric light stations; for waterworks the Wisc. R. R. Comm. uses 5 and 4.

Approved Rates used in Estimating Theoretical Depreciation (*Continued*)
By Henry Floy, *Trans. Am. Inst. E. E.*, June, 1911
(Maintenance not included)

Property	Depreciation per cent per year Straight line	Authority	Remarks
Bonds	5	Traction Val Comm.	Chicago Con. Tract. Co.
	50 per cent wearing value	Henry Floy	3d Ave Case, adopted by P. S. C., N Y.
	5	Wisconsin R R C.	
Breeching and connections	3½–10	Traction Val. Comm	Chicago Con. Tract Co.
Buildings (brick)	1½	Traction Val Comm.	Chicago Con. Tract Co
	2	E G. Connette	3d Ave Case, adopted by P S C., N. Y.
	2–4	Wisconsin R R C.	
(Frame)	2	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899
	2	St. Louis P S. C	Union Elec L. & P. Co
Cables			
Underground (high tension)	5	Henry Floy	3d Ave Case, adopted by P S C., N. Y.
Underground (low tension).	50 per cent maintenance cost	Henry Floy	3d Ave Case, adopted by P S C., N Y.
(Aerial lead covered)	$6\frac{2}{3}$	Wisconsin R R C.	
(Underground lead covered)	4	Wisconsin R. R. C.	
(Underground lead covered)	5	St Louis P. S. C.	Union Elec L & P. Co
Coal and ash handling machinery	7	Traction Val. Comm	Chicago Con. Trac. Co
	5	Henry Floy	3d Ave. Case, adopted by P. S C, N. Y.
	10	Wisconsin R R. C.	
Condensers	4	Traction Val. Comm	Chicago Con Tract Co
	5	B J. Arnold	Coney Island & Brooklyn, adopted by P S C, N.Y
	5	Henry Floy	3d Ave Case, adopted by P. S. C, N. Y.
	5	Wisconsin R. R C	
	10	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899
	$6\frac{2}{3}$	St. Louis P. S. C.	Union Elec. L. & P. Co
Conduits.	1	Henry Floy	3d Ave Case, adopted by P. S. C., N.Y.
	2	Wisconsin R R C.	
	2	St. Louis P. S C.	Union Elec L & P. Co.
Cross arms	$8\frac{1}{3}$–$12\frac{1}{2}$	Wisconsin R R C	
Engines (Steam)	3–5	Traction Val. Comm	Chicago Con Traction Co.
(Steam)	5–$7\frac{1}{2}$	B J Arnold	Coney Island & Brooklyn, adopted by P. S.C, N.Y.
(Steam)	5	Henry Floy	3d Ave Case, adopted by P. S C., N.Y.
(Gas)	$6\frac{2}{3}$	Wisconsin R. R C	
(Steam, slow speed)	5	Wisconsin R R C	
(Steam, high speed).	$6\frac{2}{3}$	Wisconsin R. R C	
	5	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899
	$6\frac{2}{3}$	St. Louis P S C	Union Elec. L & P Co.
Feeders (W. P Insulation)	Dependent on observed wear	Traction Val Comm	Chicago Con Trac. Co.
	$6\frac{1}{4}$	Wisconsin R R C	
Foundations, Machinery	Same as life of apparatus supported	Traction Val. Comm.	Chicago Con. Trac Co.
	Same as life of apparatus supported	Henry Floy	3d Ave. Case, adopted by P. S. C, N. Y.

Approved Rates used in Estimating Theoretical Depreciation *(Continued)*

By Henry Floy, *Trans. Am. Inst E. E.*, June, 1911

(Maintenance not included)

Property	Depreciation per cent per year Straight line	Authority	Remarks
Fuel oil handling machinery	4	Traction Val Comm	Chicago Con. Tract. Co
Generators	3–8	Traction Val. Comm	Chicago Con Tract Co.
	5	B. J. Arnold	Coney Island & Brooklyn, adopted by P S C, N Y.
	5	Henry Floy	3d Ave Case, adopted by P. S C, N. Y.
(Modern type)	5	Wisconsin R. R C.	
(Obsolete type)	6⅔	Wisconsin R. R. C	
(Steam turbo)	5	Wisconsin R R. C.	
	10	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899.
	6⅔	St Louis P. S C.	Union Elec. L. & P Co
Heaters	4–6	Traction Val Comm	Chicago Con. Tract Co.
(Feed water, closed)	3⅓	Wisconsin R R C	
(Feed water, open).	3⅓	Wisconsin R. R C	
Meters:			
(Electric switchboard)	5	Wisconsin R. R. C.	
(Electric service)	6⅔	Wisconsin R R. C.	
(Electric)	8	St Louis P S C.	Union Elec. L & P. Co.
Motors (Railway)	3⅓	Traction Val. Comm	Chicago Con. Tract. Co.
(Railway)	By inspection	B J. Arnold	Coney Island & Brooklyn, adopted by P S.C, N Y.
(Railway)	5	Henry Floy	3d Ave. Case, adopted by P. S. C., N. Y.
	5	Wisconsin R. R. C.	
(Railway)	10	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899.
Paving	50 per cent wearing value	B. J Arnold	Coney Island & Brooklyn, adopted by P S.C, N Y.
	50 per cent.	Henry Floy	3d Ave. Case, adopted by P. S. C, N. Y.
Piping and covering.	4–4½	Traction Val. Comm	Chicago Con Trac Co.
	6	B. J. Arnold	Coney Island & Brooklyn, adopted by P S C, N Y.
	5	Henry Floy	3d Ave. Case, adopted by P S C, N. Y.
	5	Wisconsin R. R. C.	
	5	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899
	6⅔	St. Louis P. S C.	Union Elec L & P Co
Poles (Steel)	2	Henry Floy	3d Ave Case, adopted by P. S. C, N. Y.
(Wood in concrete)	5	Wisconsin R. R. C	
(Wood in earth)	5½–8⅓	Wisconsin R. R. C	
(Iron)	2½	Wisconsin R. R. C.	
(Wooden)	10	Arbitrators	Street Lighting Controversy, Atlanta, Ga, 1899
Pumps	5	Traction Val. Comm.	Chicago Con. Tract. Co.
	5	B. J Arnold	Coney Island & Brooklyn, adopted by P S C., N Y.
	5	Henry Floy	3d Ave Case, adopted by P. S. C., N Y.
(Small steam)	6⅔	Wisconsin R. R C.	
	5	Arbitrators	Street Lighting Controversy, Atlanta, Ga., 1899.
	6⅔	St. Louis P. S C.	Union Elec L. & P. Co.
Rolling Stock:			
(Open car bodies)	4	Traction Val. Comm	Chic. Con. Tract Co.
(Open trailer bodies)	4	Traction Val Comm.	Chic. Con. Tract Co.
(Closed car bodies)	5	Traction Val. Comm.	Chic Con. Tract Co.
(Trucks)	3⅓	Traction Val Comm.	Chic. Con. Tract Co
(Closed and open cars)	5	B. J Arnold	Coney Island & Brooklyn, adopted by P.S C., N.Y.
(Trucks)	5	B. J. Arnold	Coney Island & Brooklyn, adopted by P.S C., N.Y.
	5	Henry Floy	3d Ave Case, adopted by P. S. C., N. Y.
(Car bodies and equipment)	6⅔	Wisconsin R. R. C.	
Stack	3	Tract. Val. Comm.	Chicago Con. Tract. Co.
(Steel)	10	B. J. Arnold	Coney Island & Brooklyn adopted by P.S C., N.Y.

Approved Rates used in Estimating Theoretical Depreciation (*Continued*)
By Henry Floy, *Trans. Am. Inst. E. E.*, June, 1911
(Maintenance not included)

Property	Depreciation per cent per year Straight line	Authority	Remarks
Stokers.			
(Fixed parts)	5	Tract. Val Comm	Chicago Con. Tract Co
(Moving parts)	20	Tract Val Comm	Chicago Con Tract. Co
Storage batteries	5	Henry Floy	3d Ave Case adopted by P. S. C., N. Y.
	6⅔	Wisconsin R. R. C	
	5	St. Louis P. S. C.	Union Elec. L. & P. Co
Switchboard and wiring	3	Traction Val. Comm	Chicago Con. Tract Co.
	6	B. J. Arnold	Coney Island & Brooklyn, adopted by P.S C., N Y.
	5	Henry Floy	3d Ave Case, adopted by P. S. C., N. Y.
(Modern type)	5	Wisconsin R. R. C.	
(Obsolete type)	6⅔	Wisconsin R. R C	
	8	St Louis P S C.	Union Elec L. & P. Co.
Telephones	10	Wisconsin R. R C.	
Track (Rail joints)	5	Tract Val. Comm	Chicago Con. Tract. Co
(Ties)	5	Tract. Val. Comm	Chicago Con. Tract Co.
(Rails)	Dependent on observed wear	Tract. Val. Comm.	Chicago Con. Tract Co.
(Special work)	Dependent on observed wear	Tract. Val. Comm.	Chicago Con. Tract. Co
(Straight and special work)	50 per cent wearing value	B J Arnold	Coney Island & Brooklyn, adopted by P.S C., N Y
(Straight and special work).	50 per cent wearing value	Henry Floy	3d Ave Case, adopted by P. S. C., N. Y.
(Special work)	8¼	Wisconsin R. R C	
(Straight track)	5½	Wisconsin R. R C	
Transformers.			
(Station service)	5	Wisconsin R R. C.	
	6⅔	Wisconsin R. R C.	
	6⅔	St. Louis P. S C.	Union Elec L. & P. Co
Turbines:			
(Steam)	5	Wisconsin R. R. C.	
(Water)	3⅓	Wisconsin R. R. C	
(Steam)	6⅔	St Louis P. S. C.	Union Elec L. & P Co
Wire.			
Trolley	Allowance of 80 5 lb per 1000 ft. for wearing value of No 0 wire	Tract Val Comm.	Chicago Con. Tract. Co.
Trolley	Allowance of 106.8 lb. for No. 00	Tract. Val. Comm.	Chicago Con. Tract. Co.
Trolley	From observation	B. J. Arnold	Coney Island & Brooklyn, adopted by P.S C., N. Y.
Trolley	No 0, under 1 min. headway 50 per cent.	Wisconsin R. R. C.	
Trolley	No 00, under 1 min. headway 40 per cent.	Wisconsin R. R. C.	
Trolley	No. 000, under 1 min. headway 33⅓ per cent.	Wisconsin R. R. C.	
Weatherproof	50 per cent. maintenance cost.	Henry Floy	3d Ave. Case, adopted by P. C. S., N. Y.
Weatherproof	6¼ per cent	Wisconsin R. R. C.	
Weatherproof	7½ per cent.	Arbitrators	Street Lighting Con , Atlanta, Ga., 1899.

In the Milwaukee Three-cent Fare case, a number of engineers testified as to rates of depreciation, and these have been tabulated in comparison with rates specified by other engineers and companies. The table follows:

Rates of Depreciation Compared for the Milwaukee Case

	Cooley-Beggs-Starret-Pence				Chicago		Officials	Arnold
	Milwaukee three-cent case				Union T. Co case	U. T Co Stone & Webster	M. E R & L Co	Chicago appraisals 4 cases
Buildings:								
Improvements	1 & 2		2	2	2	2	3	•
Track, main.	8	8.5	8		7 75	7 2	7 5	RI wear Sub –5
Special work.				12 5–7	7 75	7 2	12	10 years
Paving	10	8.5	8		10–4	10–4	10	10–25 Grt.
Cars, bodies and trucks	8	6 66	6	6 66–5	5	5	7 5	As shown
Electrical equipment		8 5			8.5–6⅔	8 5–6⅔	6	12–25–30
Electrical distribution	8	8 5	7				7 5	
Poles—iron				2.5	5	5		2 5
Poles—wood				8⅓ – 6⅔			7.5	
Overhead equipment.					14–10	14–10		20 total
Wiring, etc.				4			5	1½
Feeders and cables								2
Power plant	8		5					2
Cranes				6 66–5	6.66	5	5	3–5
Engines				8.5–6⅔	6.66	5	7 5	3–½
Boilers								Steel–7
Stacks								Brick–3 25 total
Stokers								5–10
Economizers								10–steel
Breechings								5
Pumps				5	6 66	5	7.5	3½
Piping and covering								3
Heaters				20	6.66	5	10	7
Coal and ash machinery				5	6 66	5	7 5	5
Generators				2	6 66	6	5	2
Switchboards					6 66	5	10	
Storage batteries								6
Foundations								3–10
Storage bins								
Shop tools and machinery	8		7	10–3⅓	5	2	7.5 5	As shown As shown
Furniture and fixtures	8		5				7 5	
Telephone system								As shown
Horses and wagons	10		5					5
Miscellaneous								

The following table shows the rates used by Marwick, Mitchell & Co., a prominent firm of accountants, in the appraisal of a large street railway system.

Rates of Depreciation, Exclusive of Maintenance Charge

Office furniture and fixtures	5.0 per cent.
Buildings	2.5 per cent.
Wagons and horses	10.0 per cent.

Roadway:	
Rails in city streets.	4.5 per cent
Rails in country roads	3 5 per cent.
Rails in private right-of-way	3 5 per cent.
Substructure in city streets	4.5 per cent.
Substructure in country roads	7 0 per cent.
Substructure in private right-of-way	10 0 per cent.
Special work	9 0 per cent.
Paving:	
Block	2.5 per cent.
Asphalt	7.0 per cent.
Brick, macadam, etc	4.5 per cent.
Tracks in car houses	3 5 per cent.
Fences:	
Snow	10 0 per cent.
Other	5 0 per cent.
Power house and substation equipment:	
Engines Boilers Heaters Conveyors Pumps and auxiliaries	4.5 per cent
Rotaries Transformers Switchboards Auxiliaries	4 5 per cent.
Storage batteries	3 0 per cent
Turbines and generators	6 0 per cent.
Rolling stock:	
Cars (including all equipment)	5 0 per cent.
Utility equipment	5 0 per cent.
Shop machinery:	
Machine tools	5 0 per cent.
Hand tools.	10.0 per cent
Line:	
Poles—iron	2.0 per cent.
Poles—wood	7.0 per cent.
Wire—trolley	5.0 per cent.
Wire—guard	10.0 per cent.
Spans with ears, etc.	5.0 per cent.
Conduits and manholes	2.0 per cent.
Feeder cables, overhead	3.0 per cent.
Feeder cables leaded, in ducts	4.0 per cent.
Transmission lines.	2.0 per cent.
Special work	8.0 per cent.
Signals, telegraph, etc.	10.0 per cent.
Bridges	2.5 per cent.

Rates of Depreciation Compared for the Milwaukee Case

	Cooley-Beegs-Starret-Pence				Chicago		Officials	Arnold
	Milwaukee three-cent case				Union T Co case	U T Co Stone & Webster	M E R & L Co	Chicago appraisals 4 cases
Buildings:								
Improvements	4 & 2		2	2	2	2	3	
Track, main	8	8 5	8		7 75	7 2	7.5	Rl. wear Sub. –5
Special work				12 5–7	7 75	7 2	12	10 years
Paving	10	8 5	8		10–4	10–4	10	10–25 Grt.
Cars, bodies and trucks	8	6 66	6	6.66–5	5	5	7 5	As shown
Electrical equipment		8 5			8 5–6⅔	8 5–6⅔	6	12–25–30
Electrical distribution	8	8 5	7				7 5	
Poles—iron				2 5	5	5		2 5
Poles—wood				8⅓ – 6⅔			7 5	
Overhead equipment								20 total
Wiring, etc					14–10	14–10		
Feeders and cables				4			5	1½
Power plant	8		5					2
Cranes								2
Engines				6 66–5	6 66	5	5	3–5
Boilers				8.5–6⅔	6 66	5	7.5	3–½
Stacks								Steel–7 Brick–3
Stokers								25 total
Economizers								5–10
Breechings								10–steel
Pumps								5
Piping and covering				5	6 66	5	7 5	3½
Heaters								3
Coal and ash machinery				20	6 66	5	10	7
Generators				5	6 66	5	7 5	5
Switchboards				2	6 66	6	5	2
Storage batteries					6 66	5	10	
Foundations								6
Storage bins								3–10
Shop tools and machinery	8		7	10–3⅓	5	2	7 5	As shown
Furniture and fixtures	8		5				5	As shown
Telephone system							7 5	
Horses and wagons	10		5					As shown
Miscellaneous								5

The following table shows the rates used by Marwick, Mitchell & Co., a prominent firm of accountants, in the appraisal of a large street railway system.

Rates of Depreciation, Exclusive of Maintenance Charge

Office furniture and fixtures	5.0 per cent.
Buildings	2.5 per cent.
Wagons and horses	10.0 per cent.

Roadway:	
Rails in city streets	4 5 per cent
Rails in country roads	3.5 per cent.
Rails in private right-of-way	3 5 per cent.
Substructure in city streets	4.5 per cent.
Substructure in country roads	7.0 per cent.
Substructure in private right-of-way	10 0 per cent
Special work	9 0 per cent.
Paving:	
Block	2.5 per cent.
Asphalt	7 0 per cent.
Brick, macadam, etc.	4 5 per cent.
Tracks in car houses	3 5 per cent.
Fences:	
Snow	10.0 per cent.
Other	5 0 per cent.
Power house and substation equipment:	
Engines Boilers Heaters Conveyors Pumps and auxiliaries	4 5 per cent.
Rotaries Transformers Switchboards Auxiliaries	4.5 per cent.
Storage batteries	3 0 per cent.
Turbines and generators	6.0 per cent.
Rolling stock:	
Cars (including all equipment)	5.0 per cent.
Utility equipment	5.0 per cent.
Shop machinery:	
Machine tools	5.0 per cent.
Hand tools	10 0 per cent.
Line:	
Poles—iron	2.0 per cent.
Poles—wood	7.0 per cent.
Wire—trolley	5.0 per cent.
Wire—guard	10.0 per cent.
Spans with ears, etc.	5.0 per cent.
Conduits and manholes	2.0 per cent.
Feeder cables, overhead	3.0 per cent.
Feeder cables leaded, in ducts	4.0 per cent.
Transmission lines	2 0 per cent.
Special work	8.0 per cent.
Signals, telegraph, etc.	10.0 per cent.
Bridges	2.5 per cent.

The two tables following have been compiled by Mr. George W. Cravens and carry the comparison still farther.

Depreciation

By George W. Cravens, *Electrical Review*, April 23, 1910

Table I.—Percentage Depreciation per Annum

Items	Chicago traction commission	Chicago Union Traction Co.	Milwaukee Electric Railway & Light Co.	Wisconsin railroad Commission	Average English	Average Scotch Practice	Philip Dawson	Stone and Webster	Industrial power plants	Professor G. F. Gebhardt	Miscellaneous sources
Buildings	. .	2	3	2	2 5	2 5	1–2	2	1–2	2	2
Boilers.. ..	3.5–10	6.6	7 5	6.6–8.5	5	5	8–10	5	2.5–3 3	4–6 6	7 5
Steam piping.	3.5	6 6	7.5	5	5	5	8–10	5	2 5–3.3	5–8	5
Auxiliaries . ..	5–10	6.6	5	6 6–8 5	5	5	8–10	5	4–6.6	3–5	7 5
Steam engines .	3–10	6.6	5	5–6.6	5	5	4–6	5	2 5–5	4–6 6	5
Steam turbines. .	......	.....	5	. . .	5	5	7–9	5	2 5–5	4	4
Belted generators.	5–10	6.6	7.5	5	5	5	5–10	5	6.6	3.3–4	7.5
Direct connected generators....	5	6.6	7.5	5	5	5	4–8	5	4	3.3–4	5
Wires and cables..	2	6.6	5	2	5	3	3–5	5	4–6.6	5	5
Switchboards, etc..	2	6.6	5	2	7.5	5	8–10	5	2–5	...	5
Rotary converters.	. ....	6.6	5	5	5	5	8–10	5	4–5	4	5
Transformers. ..	. . .	6.6	5	5	5	5	5–6	5	3–5	4	5
Motors... .	5–10	6.6	5	5	5	5	5–8	5	4–6.6	5	5
Storage batteries..	... .	6.6	10	.....	5	5	9–11	5	5–10	6 6	10
Overhead systems	. . .	10–14	7.5	..	...	3	4–8	10–14	5–10	...	10
Cars.....	..	5–8 5	6-7.5	5–6.6	10	7.5	4–6	5–8.5	. . .	.	7 5
Trackwork. . ..	...	7.75	7.5–12	5	5	8	7–13	7 2	.	..	7 5
Shop equipment...	3–10	5	7.5	3.3–10	7.5	7.5	12–15	5	4–10	...	7.5
Supplies and miscellaneous......	5	.....	5	....	5	7.5	1 5–2	. ...	5	. . .	5

Table II.—Percentage Depreciation for Given Service

Items of equipment	Light or intermittent service	Heavy or continuous service
Boilers, water tube..	4–6.6	5–8.3
Boilers, fire tube	5–6.6	6.6–10
Piping, steam and water.	4–5.5	5.5–8.3
Auxiliaries, steam.	3–5	4–6 6
Engines, steam.	4–5	5–6.6
Turbines, steam.	3–4	4–5
Generators, belted.	4–6.6	5–8 3
Generators, direct connected..	3–5	4–6.6
Wires and cables.	3–5	4–6 6
Switchboards and instruments.	2–5	5–8 3
Rotary converters.	3–5	4–6.6
Transformers.	3–5	4–6 6
Motors, A. C. and D. C.	4–6 6	5–8.3
Storage batteries.	5–6 6	6 6–10
Overhead systems.	4–10	8.3–12 5
Cars and equipments.	5–8.3	6.6–10
Trackwork, ballast, etc...	5–8.3	7.5–12.5
Shop equipment, tools, etc.	5–10	7.5–15

The Chicago Telephone Commission in 1908 gave the following table of life, depreciation and salvage.

Life and Depreciation of Telephone Wires

Property	Life in years	Per cent. to depreciation account	Per cent. salvage	Per cent. to reconstruction and insurance
Conduit, main, clay in concrete	50	0 89	0	1½
Conduit, main, fiber in concrete	20	3 72	0	1½
Conduit, subsidiary	20	3 72	0	2
Cable, main	20	3.72	.	2
Cable, subsidiary	15	5 38	40	2
Aërial cable	15	5 38	.. .	3
Poles, including cross arms	10	8 73	0	4½
Aerial strand	12	7 05	0	3½
Aërial cable, terminals	12	7 05	0	3
Aërial wire, copper	15	5 38	70	3
Drop wires	8	11.25	15	4
Subscribers' station instruments	10	8 73	5	2
Private branch exchange switchboards	8	11 28	20	2
Central office switchboards	8	11 25	20	2
Buildings, fireproof	40	1 33	0	1
Teams, tools, furniture, etc	4	23 92	10	0

The following table is one compiled by "Data" from various sources showing life and rates chargeable for depreciation on telephone plants.

Telephone Plant, Life and Depreciation of, from "Data."[1] Annual Depreciation—Per Cent. of First Cost in Place

Class of plant	Life in years	Junk value per cent. of first cost in place	Without cumulative interest	With 2 per cent interest	With 3 per cent interest	With 6 per cent. interest
Buildings:						
Fireproof	25–50	25	3.0–1.5	2.3–0.9	2.1–0 7	1 4–0.25
Mill construction	25	20	3.2	2.8	2.2	1.5
Cables:						
Aërial exchange	12–15	30	5.8–4.7	5.2–4.0	4 9–3.8	4.1–3.0
In buildings	10	30	7 0	6 4	6.1	5.3
Underground	.. ..	....	......	. ...	. .....	..... .
Exchange, main	20–25	40	3.0–2 4	2 5–1 9	2 2–1.6	1.6–1.1
Exchange, subsidiary	13–20	40	4.6–3.0	4 1–2.5	3 8–2.2	3.2–1 6
Toll	30	40	2 0	1.5	1 3	0.76
Submarine	10	0	10 0	9.1	8 7	7 6
Cable terminals	10–12	0	10.0–8.3	9.1–7.5	8.7–7.0	7.6–5.9
Conduits, main[2]						
Clay vitrified	50	0	2.0	1.2	0.89	0 34
Concrete	50	0	2 0	1.2	0.89	0 34
Fiber	20	0	5.0	4.1	3.7	2.7
Iron	20	0	5 0	4.1	3 7	2.7
Wood, creosoted	20	0	5.0	4.1	3.7	2 7
Conduits, subsidiary	15	0	6 7	5.8	5 4	4.3
Furniture and fixtures	10	0	10.0	9.1	8.7	7.6
Poles:						
Cedar, 35 feet and under	10–15	0	10 0–6.7	9 1–5 8	8.7–5 4	7.6–4 3
Cedar, over 35 feet	15–20	0	6 7–5.0	5.8–4.1	5.4–3.7	4 3–2 7

[1] Adapted from practice of A. T. & T. Co., Wisconsin R. R. Commission and others

[2] Includes manholes.

Telephone Plant, Life and Depreciation of, from "Data." Annual Depreciation—Per Cent. of First Cost in Place (*Continued*)

Class of plant	Life in years	Junk value per cent. of first cost in place	Without cumulative interest	With 2 per cent interest	With 3 per cent interest	With 6 per cent interest
Chestnut, 35 feet and under	8–12	0	12 5–8 3	11.6–7.5	11 2–7.0	10 1–5 9
Chestnut, over 35 feet	12–15	0	8 3–6.7	7.5–5 8	7 0–5 4	5.9–4 3
Pine, creosoted	20	0	5 0	4 1	3 7	2.7
Poles and cross arms, average exchange.	10	0	10.0	9 1	8.7	7 6
Poles and cross arms, average toll.	15	0	6 7	5 8	5 4	4 3
Power plants:						
Switchboards, central office	8–10	15	10 6–8 5	9 9–7 8	9 6–7 4	8 6–6 4
Subscribers' station equipment						
Instruments	8–10	10	11 2–9.0	10.5–8.2	10.1–7.9	9 1–6 8
P B. X switchboards	10	10	9 0	8 2	7 9	6 8
Tools and teams.	5	0	20 0	19 2	18 8	17 7
Wire exchange lines						
Copper, bare.	10–15	40	6 0–4 0	5 5–3 5	5.2–3 2	4 5–2 6
Copper insulated tw pr	10	10	9 0	8.2	7 8	6 8
Iron, bare	8–10	0	12 5–10 0	11 6–9 1	11 2–8.7	10 1–7 6
Wire toll lines:						
Copper, bare.	40	40	1 5	1.0	0 8	0 39
Iron, bare.	15	0	6 7	5 8	5 4	4 3

U. S. Government Allowance for Depreciation

Buildings:	Per cent. per annum
Brick, occupied by owner	1 to 1–1/4
Brick, occupied by tenant	1–1/4 to 1–1/2
Frame, occupied by owner	2 to 2–1/2
Frame, occupied by tenant	2–1/2 to 3

Life of Items of Physical Plant

Following is a good average estimate of the life of the different items going to make up the physical plant of the following named properties:

Waterworks.
Power plants.
Buildings.
Electric light and railways.
Telephones.
Gas plants.

Some of the items have been copied directly from a "Memorandum regarding values used for lives of structures and equipment," gotten out by the Joint Engineering Department of the Wisconsin Commissions.

Waterworks:	Years' life
Wells, driven or drilled	50 to 75
Wells, large open, stone or brick walled	75 to 100
Suction piping and intakes	30 to 50
Stand pipes	30 to 50
Reservoirs	50 to 100
Filter beds	30 to 50

Distribution System, for Waterworks:	
Cast-iron mains, including fittings and valves	75 to 100
Wrought-iron mains and services galvanized and including valves and fittings	30 to 50
Cast-iron mains and services, black	25 to 35
Services to buildings, lead	50 to 100
Power Plant Equipment:	
Gas engines	10 to 15
Corliss slow speed engines	25 to 30
High speed engines	15 to 20
Waterworks, pumping engines, duplex	20 to 25
triplex	20 to 30
flywheel	30 to 40
Boiler feed pumps	15 to 20
Geared power pumps	20 to 30
Centrifugal pumps	20 to 30
Boilers, fire tube	10 to 15
Boilers, water tube	20 to 30
Heaters	20 to 30
Condensers	20 to 30
Coal and ash conveyers	10
Turbine waterwheels, built before 1900	25 to 40
Turbine waterwheels, built after 1900	30 to 50
Jack shaft	20 to 40
Piping	20 to 30
Pipe covering	same as pipe
Leather belting	20 to 25
Generators, motors and rotaries:	
modern	20
obsolete	15
Static transformers, including regulators and compensators, station type	20
Turbo-generators	20
Switchboard and wiring complete:	
modern type	20 to 30
obsolete type	15 to 20
Switchboard and instruments:	
modern type	20 to 30
obsolete type	15 to 20
Storage batteries	10 to 15
Lightning arresters, station type	15 to 20

Foundations for machinery give same life as the machine they support.

Buildings (Wisconsin Commissions):	
First class stone and brick (office buildings)	75
Second class shops, car barns and power stations	50
Brick gas retort houses	30
Frame dwellings	35
Frame stables and coal sheds	20 to 25

Electric Light Railways:

Copper wire, weatherproof insulation	10 to 15
Cable, lead covered, aërial	10 to 15
Cable, lead covered, underground, in conduits	20 to 25
Conduit	30 to 50
Manholes	30 to 50
Poles, cedar, in earth	10 to 18
Poles, cedar, in concrete	12 to 20
Poles, iron, in concrete	15 to 30
Pole anchors and guys	10 to 20
Cross arms	10 to 15
Service transformers	10 to 15
Fuse boxes	10 to 12
Lightning arresters	10 to 12
Arc lamps and span equipment	10 to 15
Nernst lamps	8 to 10
Service wattmeters	10 to 15
Railway spans and bracket equipment	15 to 20

Trolley wire; life depends on headway; should never be worn to less than 75 per cent. of its original size.

No. 0 wire at one minute headway, life two years.
No. 00 wire at one minute headway, life two and one-half years.
No. 000 wire at one minute headway, life three years.

Telephone Equipment (Wisconsin Commissions):

Central office equipment including distributing frame	10
Power, plant and wire	8
Poles	12 to 15
Cross arms	8 to 12
Iron wire	8 to 15
Copper wire	20
Weatherproof iron wire	15
Cables, lead covered, aërial	12
Cables, lead covered, underground	20
Subscribers' sets	10
Furniture and tools	7

Gas Plants (Wisconsin Commissions):

Benches	25
Scrubbers and condensers	30
P. and A. tar extractors	40
Cast-iron washers	40
Exhausters	25
Purifiers, modern	50
Cast-iron gas connections, within the plant	50
Station meter cases	50
Station meter drums	20
Ammonia concentrators	15
Ammonia storage tanks, wrought iron or steel	15
Tar and ammonia wells	50
Water gas machines complete	30

Centrifugal blowers	15
Holders	50
Governors	50
Cast-iron mains. 3 inches and 4 inches	50
Cast-iron mains. 6 inches and larger	75
Wrought-iron and steel mains under 3 inches	20
Wrought-iron and steel mains over 3 inches	30
Services	20
Consumers' meters	25
Consumers' governors	25
Retort house floors, depending on construction	15 to 30

Foundations for machinery, same life as machine they support.

Waterworks Property, Limits of Useful Life

By Leonard Metcalf, p. 24, vol. 64, *Trans. Am. Soc. C. E.*, 1909

	Useful life
Reservoirs[1]	50 to 100 years
Standpipes	25 to 40 years
Masonry buildings	40 to 50 years
Wooden buildings	20 to 50 years
Cast-iron pipe of large diameter	50 to 75 years
Cast-iron pipe of small diameter	20 to 40 years
Steel pipe	25 to 50 years
Wood stave pipe	20 to 30 years
Wrought-iron service pipe	15 to 30 years
Meters	20 to 30 years
Hydrants	40 to 50 years
Gates	40 to 50 years
Pumping and auxiliary machinery	20 to 30 years
Steam engines	15 to 25 years
Boilers	12 to 16 years
Electrical machinery	20 to 30 years

[1] Except where subject to heavy deposit of silt.

APPENDIX D

REFERENCES TO PUBLISHED COST DATA AND METHODS OF ESTIMATING

It is hoped that the references below will be of material help to the student and to the engineer who has occasion to estimate cost of unfamiliar work. It is realized that the list of references is not as complete as it might well be. Notification of good additional references on cost data and estimating methods will be gladly received by the present writer.

Suggestions to Inexperienced Estimators on the Use of Cost Data.—(1) Visualize the methods and conditions which promise to obtain on the proposed work for which you are estimating. (2) Note in the following list the names of those publications which give promise of yielding cost data useful for the estimating in hand. (3) From the available publications thus noted, select such groups of cost data as appear to be of use in the estimating in hand. (4) Judge the dependability of each such group, basing the judgment on the reputation of the author of the data and on the completeness and consistency of the data. (Read §91.) (5) Select from the more dependable groups those which pertain to work the methods and conditions of which more nearly approximate the anticipated methods and conditions of the proposed work. (6) Note the differences between the methods and conditions of each group of cost data and the anticipated methods and conditions of the proposed work. (7) For each group, judge the effect of the said differences on the cost data, and modify the cost data accordingly. (8) Weigh the modified groups of data; and, according to judgment, adopt one set or another, or some compromise, and use the adopted set for the estimating in hand.

1. REFERENCES TO COST DATA

"Handbook of Cost Data," H. P. Gillette, Chicago, Myron C. Clark Publishing Co., third edition, 1914, 1854 pages, $5.

"Radford's Estimating and Contracting," Wm. A. Radford, Chicago, the Radford Architectural Company, 1913, 887 pages, $2.

"Civil Engineer's Pocket Book," John C. Trautwine, Philadelphia, The Trautwine Company, nineteenth edition, 1914, 1300 pages, $5. (Look in the index of said book under "cost" and under "price list.")

"Civil Engineers' Pocket Book," Albert I. Frye, New York, D. Van Nostrand Co., 1913, 1611 pages, $5. (Look in the index of said book under the name of the thing for which cost data are desired.)

"American Civil Engineer's Pocket Book," Mansfield Merriman, Editor-in-chief, New York, John Wiley & Sons, second edition, 1914, 1380 pages, $5. (Look in the index of said book under the name of the thing for which cost data are desired.)

"Handbook of Electrical Cost Data," H. A. Foster, New York, D. Van Nostrand Co., 1914.

"Architects' and Builders' Pocket Book," F. E. Kidder, New York, John Wiley & Sons, fifteenth edition, 1908, 1661 pages, $5. (Look in the index of said book under "cost.")

"The New Building Estimator," Wm. Arthur, New York, David Williams Co., third edition, 1910, 500 pages, $2.50.

"Concrete Costs," F. W. Taylor and S. E. Thompson, New York, John Wiley & Sons, 1912, 709 pages, $5.

"Data," Chicago, Myron C. Clark Publishing Co., $2 per annum.

"Leafax," Philadelphia, Standard Corporation, $1 per annum.

Engineering and Contracting (devoted to the economics of Civil Engineering design and to methods and cost of construction), Chicago, $2 per annum. (Look in each volume index under "cost.")

Transactions American Institute of Electrical Engineers. (Look under "cost" in the topical index of "Index to the Transactions of American Institute of Electrical Engineers," vol. 1 (1884 to 1900) and vol. 2 (1901 to 1910).)

Transactions American Institute of Mining Engineers. (Look in the indices under "cost.")

Engineering and Mining Journal, New York, $5 per annum. (Look in the indices under "cost.")

Engineering Record, New York, $3 per annum. (Look in each volume index under "cost.")

Engineering News, New York, $5 per annum. (Look in each volume index under "cost.")

"Earthwork and Its Cost," H. P. Gillette, New York, McGraw-Hill Book Co., second edition, 1912, 238 pages, $2.

"Rock Excavation (Methods and Cost)," H. P. Gillette, Chicago, Myron C. Clark Publishing Co., 1907, 376 pages, $3.

"Dredges and Dredging," C. Prelini, New York, D. Van Nostrand Co., 1911, 279 pages, $3.

"Handbook of Construction Plant," R. T. Dana, Chicago, Myron C. Clark Publishing Co., 1914, 702 pages, $5.

"Excavating Machinery," A. B. McDaniels, New York, McGraw-Hill Book Co., 1913, 340 pages, $3.

"Concrete Construction (Methods and Cost)," H. P. Gillette and C. S. Hill, Chicago, Myron C. Clark Publishing Co., 1908, 690 pages, $5.

"A Treatise on the Design and Construction of Mill Buildings and Other Industrial Plants," H. G. Tyrrell, Chicago, Myron C. Clark Publishing Co., 1911, 490 pages, $4. (See pp. 38–68 and 410–429.)

"Railroad Location Surveys and Estimates," F. Lavis, Chicago, Myron C. Clark Publishing Co., 1908, 270 pages, $3. (Chapter VIII, Estimates and Tables of Quantities.)

"Railroad Structures and Estimates," J. W. Orrock, New York, John Wiley & Sons, 1909, 270 pages, $3.

"Railroad Construction," C. L. Crandall and F. A. Barnes, New York, McGraw-Hill Book Co., 1913, 320 pages, $3.

"Handbook for Highway Engineers," W. G. Harger and E. A. Bonney, New York, McGraw-Hill Book Co., 1912, 493 pages, $3. (Chapter X on cost data and estimates.)

"A Practical Treatise on Sub-aqueous Foundations," C. E. Fowler, New York, John Wiley & Sons, 1914, 814 pages, $7.50.

"Rock Drilling," R. T. Dana and W. L. Saunders, New York, John Wiley & Sons, 1912, 319 pages, $4.

"Practical Shaft Sinking," Francis Donaldson, New York, McGraw-Hill Book Co., 1910, 143 pages, $2.

"Modern Tunneling (with Special Reference to Mine and Water-supply Tunnels)," D. W. Brunton and J. A. Davis, New York, John Wiley & Sons, 1914, 450 pages, $3.50.

"Modern Tunnel Practice," D. McN. Stauffer, New York, McGraw-Hill Book Co., 1906, 314 pages, $3.50. (Chapter X, Some Data on the Cost of Tunneling.)

"Index of Mining Engineering Literature," W. R. Crane, New York, John Wiley & Sons, vol. 2 (1909 to 1911, both inclusive). (Look in the subject index under "cost.")

"Mining World Index of Current Literature," G. E. Sisley, Chicago, Mining World Co., vol. 4 (last half-year, 1913) and vol. 5 (first half-year, 1914). (Look in the subject index under "cost.")

Reports of Mining Companies, published from time to time in *Engineering and Mining Journal* and other mining periodicals.

"Safety and Efficiency in Mine Tunneling," Bulletin 57, Bureau of Mines, Department of the Interior, Washington, D. C., D. W. Brunton and J. A. Davis, 1914.

"Cost of Mining (An Exhibit of the Results of Important Mines throughout the World)," J. R. Finlay, New York, McGraw-Hill Book Co., 1909, 415 pages, $5.

"Smelter Construction Costs," E. H. Jones, McGraw-Hill Book Co., 1914, 152 pages, $2.

"Telephone Line Construction—Methods and Cost," Clarence Mayer, Chicago, Myron C. Clark Publishing Co., 1910, 300 pages, $4.

"Logging," R. C. Bryant, New York, John Wiley & Sons, 1913, 590 pages, $3.50.

2. REFERENCES TO ESTIMATING METHODS

"Handbook of Cost Data," H. P. Gillette, Chicago, Myron C. Clark Publishing Co., third edition, 1914, 1854 pages, $5.

"Radford's Estimating and Contracting," Wm. A. Radford, Chicago, the Radford Architectural Company, 1913, 887 pages, $2.

"Earthwork and Its Cost," H. P. Gillette, New York, McGraw-Hill Book Co., second edition, 1912, 238 pages, $2.

"Estimating the Cost of Buildings," A. W. Joslin, New York, David Williams Co., 1910, 205 pages.

"Economic Theory of Railway Location," A. M. Wellington, New York, John Wiley & Sons, 1886, 980 pages, $5. (Pages 895–906 on estimation of quantities.)

"Preliminary Surveys and Estimates," T. G. Gribble, New York, Longmans, Green & Co., fifth edition, 1908, 472 pages, 7*s* 6*d*. (Chapter VIII, Graphic Calculation for Preliminary Estimates.)

"Railroad Location Surveys and Estimates," F. Lavis, Chicago, Myron C. Clark Publishing Co., 1908, 270 pages, $3. (Chapter VIII, Estimates and Tables of Quantities.)

"Handbook of Steam-shovel Work," A Report by the Construction Service Co., The Bucyrus Co., South Milwaukee, Wis., 1911, 374 pages, $1.50.

"Practical Talks on Contracting," F. G. Gilbreth and Others, Chicago, The Contractor Publishing Co., 1910, 128 pages. (Part on "Important Things to Consider in Estimating," by A. O. Davidson.)

"The Elements of Engineering Contracting," A. Suggate, London, The Technical Publishing Co., 1912, 135 pages.

"Structural Engineers' Handbook," M. S. Ketchum, New York, McGraw-Hill Book Co., 1914, 896 pages, $5. (Chapter XIII on estimates of structural steel.)

PART V

TABLES

Table

I. Simple interest formulas.
II. Compound interest formulas.
III. Formulas for sinking funds of which \$1 is one of the given elements.
IV. Formulas for sinking funds in which deposit is made at the *end* of each k-year interval.
V. Formulas for sinking funds in which deposit is made at the *beginning* of each k-year interval.
VI. Formulas for sinking funds in which deposit is made at the *beginning of each* and at the *end of the last* k-year interval.
VII. Bases of comparison and formulas for cost of service.
A. (Compound) amount of \$1 for n years.
B. Present worth of \$1 due n years hence.
C. Amount of sinking fund.
D. Principal which will produce an annuity of \$1 per year for n years.
E. Deposit required to make sinking fund amount to \$1 in n years.
F. Annuity produced for n years by a principal of \$1.
G. Days of the year.
H. Simple interest table (365-day year).
J. Simple interest table (360-day year).

Table I.—Simple Interest Formulas

Given	Required	Formula	
Principal, P Rate, i Time (years), n	Interest, I Amount, S	$I = Pin$ (Tables H and J) $S = P(1 + in)$	(1) (2)
Rate, i Time (years), n Interest, I	Principal, P	$P = I/(in)$	(3)

Table I.—Simple Interest Formulas (*Continued*)

Given	Required	Formula	
Rate, i Time (years), n Amount, S	Principal, P	$P = S/(1 + in)$	(4)
Interest, I Principal, P Rate, i	Time (years), n	$n = I/(Pi)$	(5)

See §8 and Tables H and J for examples.

Table II.—Compound Interest Formulas

Given	Required	Formula	
Rate, i Interest periods in one year, h Time (years), n Principal, \$1	Amount, s_n Interest, I	$s_n = (1 + i/h)^{hn}$ (Table A) $I = s_n - 1$	(18a) (19a)
Rate, i Interest periods in one year, h Time (years), n Amount, \$1	Present worth, p_n	$p_n = \dfrac{1}{(1 + i/h)^{hn}}$ (Table B)	(20b)
Interest periods in one year, h Time (years), n Principal, \$1 Amount, s_n	Rate, i	$i = h[(s_n)^{1/hn} - 1]$	(21)
Rate, i Interest periods in one year, h Principal, \$1 Amount, s_n	Time (years), n	$n = \dfrac{\log s_n}{h \log(1 + i/h)}$	(22)
Rate ("nominal"), i Interest periods in one year, h	"Effective" rate (compound interest on \$1 for one year), e	$e = (1 + i/h)^h - 1$	(15)
"Effective" rate, e Interest periods in one year, h	Rate ("nominal"), i	$i = h[(1 + e)^{1/h} - 1]$	(17)

USE OF TABLE III

(Table III need be used only in case the conditions of the problem are beyond the range of Tables C, D, E, and F.)

When one of the given sums—present worth, deposit, or amount—is other than $1, solve the problem first on the assumption that it is $1, and multiply the result by the number of dollars in the given sum.

Example.—What is the amount of the sinking fund in which $1 is deposited at the end of each year for 20 years, if interest is at 6 per cent. compounded semi-annually?

The formula for this case is (Eq. 31, Table III),

$$z_n = \frac{s^n - 1}{s - 1}.$$

Substituting for s its value in terms of i and h, we have

$$z_n = \frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^h - 1}.$$

In this example, $i = 0.06$, $h = 2$, and $n = 20$. Therefore

$$z_n = \frac{(1 + 0.06/2)^{2\times 20} - 1}{(1 + 0.06/2)^2 - 1} = \frac{{}^{1}1\,03^{40} - 1}{{}^{1}1.03^2 - 1}$$

$$= \frac{3.262 - 1}{1.061 - 1}$$

$$= \$37. \quad \textit{Answer}.$$

[1] 1.03^{40} is evidently the compound amount of $1 for 40 years at 3 per cent. compounded annually; and 1.03^2 is the compound amount of $1 for two years at 3 per cent. compounded annually. Therefore the values of 1.03^{40} and 1.03^2 can be taken directly from Table A.

Table III.—Sinking-fund Formulas

For cases in which one of the given elements is \$1, and the deposit is made at the *end* of each year.

n = life of sinking fund, in years.
s = amount of \$1 for one year
$= (1 + i/h)^h$.
i = rate on deposits.
h = number of interest periods in one year.

Given	Required	Formula
Deposit, \$1	Amount, z_n	$z_n = \frac{s^n - 1}{s - 1}$. (Table C) (31)
Deposit, \$1	Present worth, w_n	$w_n = \frac{s^n - 1}{s - 1} \cdot \frac{1}{s^n}$. (Table D) (32)
Deposit, \$1 Life (years), $n = \infty$	Present worth, w_∞	$w_\infty = \frac{1}{s - 1}$ (32b)
Deposit, \$1 Present worth of m-year sinking fund, w_m Present worth of $(m+n)$-year sinking fund, w_{m+n}	Present worth of n-year sinking fund deferred m years, w'_n	$w'_n = w_{m+n} - w_m$ (32c) $= \frac{s^n - 1}{s - 1} \cdot \frac{1}{s^{n+m}}$ (32d)
Amount, \$1	Deposit, d_n	$d_n = \frac{s - 1}{s^n - 1}$. (Table E) (33)
Present worth, \$1	Deposit, a_n	$a_n = \frac{(s - 1)s^n}{s^n - 1}$. (Table F) (34)
Present worth, \$1 Life (years), $n = \infty$	Deposit, a_∞	$a_\infty = s - 1$ (34b)
Deposit, \$1 Amount, z_n	Life (years), n	$n = \frac{\log [z_n(s - 1) + 1]}{\log s}$ (34c)
Deposit, \$1 Present worth, w_n	Life (years), n	$n = \frac{\log \left[\frac{1}{1 - (s - 1)w_n}\right]}{\log s}$ (34d)
Amount, \$1 Deposit, d_n	Life (years), n	$n = \frac{\log \left[\frac{s - 1}{d_n} + 1\right]}{\log s}$ (34e)
Present worth, \$1 Deposit, a_n	Life (years), n	$n = \frac{\log \left[\frac{a_n}{a_n - (s - 1)}\right]}{\log s}$ (34f)

USE OF TABLE IV

The formulas of Table IV assume that the interest period is not longer than the deposit interval; that is, the formulas do not hold when the deposit interval is shorter than the interest period. For example, Eq. 27 will give the amount of the sinking fund in which D is deposited at the end of each half-year if interest is compounded semi-annually, or quarterly, or oftener, but will *not* give the amount if interest is compounded annually.

This table need be resorted to only when the conditions of the problem are beyond the range of Tables C, D, E, and F.

Example.—If at the end of each half-year for five years, we deposit \$100 in a savings bank which pays interest at the rate of 4 per cent. compounded quarterly, what will be the accumulated amount at the end of the fifth year?

The formula to be used in this case is (Eq. 27, Table IV),

$$Z = D\frac{s^n - 1}{s^k - 1}, \quad \text{where} \quad s = (1 + i/h)^h, \quad s^n = (1 + i/h)^{hn},$$

$$s^k = (1 + i/h)^{hk}.$$

Substituting, we have

$$Z = D\frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^{hk} - 1}.$$

In this example, $D = 100$, $h = 4$, $n = 5$, and $k = 1/2$. Therefore

$$Z = 100\frac{(1 + 0.04/4)^{4\times 5} - 1}{(1 + 0.04/4)^{4\times ½} - 1} = 100\,\frac{1.01^{20} - 1}{1.01^{2} - 1} = 100\,\frac{1.220 - 1}{1.020 - 1}$$

$$= \$1100\pm. \quad \textit{Answer.}$$

Note.—1.01^{20} is the amount of \$1 for 20 years at 1 per cent. compounded annually, and 1.01^2 is the amount of \$1 for two years at 1 per cent. compounded annually. As Table A does not give amounts for the rate of 1 per cent., the values of 1.01^{20} and 1.01^2 were computed by logarithms (§14).

Table IV.—Sinking Funds

In which deposit is made at the *end* of each k-year period (§§23 to 25).

n = life of sinking fund.
k = number of years in one deposit-interval
= or $> 1/h$. For $k < 1/h$, use Eqs. 39 to 43.
s = amount of \$1 for one year
$= (1 + i/h)^h$.
i = rate.
h = number of interest periods in one year.

Given	Required	Formula
Deposit, D	Amount, Z	$Z = D\dfrac{s^n-1}{s^k-1}$ (27)
Deposit, D	Present worth, W	$W = D\dfrac{s^n-1}{s^k-1}\cdot\dfrac{1}{s^n}$ (28)
Amount, Z	Deposit, D	$D = Z\dfrac{s^k-1}{s^n-1}$ (29)
Present worth, W	Deposit, D	$D = W\dfrac{s^k-1}{s^n-1}\cdot s^n$ (30)
Deposit, D Life, n years Deferred m years	Present worth, W'	$W' = D\dfrac{s^n-1}{s^k-1}\cdot\dfrac{1}{s^{m+n}}$ (30b)
Amount, Z Deposit, D	Life, n	$n = \dfrac{\log[(Z/D)(s^k-1)+1]}{\log s}$ (30c)
Present worth, W Deposit, D	Life, n	$n = \dfrac{\log\left[\dfrac{1}{1-(W/D)(s^k-1)}\right]}{\log s}$ (30d)
Deposit, D Life, $n = \infty$	Present worth, W	$W = D\dfrac{1}{s^k-1}$ (30e)
Present worth, W Life, $n = \infty$	Deposit, D	$D = W(s^k-1)$ (30f)
Deposit, D Life, $n = \infty$ Deferred m years	Present worth, W'	$W' = D\dfrac{1}{s^k-1}\cdot\dfrac{1}{s^m}$ (30g)

USE OF TABLE V

The formulas of Table V assume that the interest period is not longer than the deposit interval; that is, the formulas do not hold when the deposit interval is shorter than the interest period. For example, Eq. 30h will give the amount of the sinking fund in which D is deposited at the end of each half-year if interest is compounded semi-annually, or quarterly, or oftener, but will *not* give the amount if interest is compounded annually.

Example.—If at the *beginning* of each half-year for five years, we deposit \$100 in a savings bank which pays interest at the rate of 4 per cent. compounded quarterly, what will be the accumulated amount at the end of the fifth year?

For this case the formula is (Eq. 30, Table V),

$Z = D\dfrac{s^n - 1}{s^k - 1}s^k$, where $s = (1 + i/h)^h$, $s^n = (1 + i/h)^{hn}$, and $s^k = (1 + i/h)^{hk}$. Substituting, we have

$$Z = D\frac{(1 + i/h)^{hn} - 1}{(1 + i/h)^{hk} - 1}(1 + i/h)^{hk}.$$

In this example, $D = 100$, $h = 4$, $n = 5$, and $k = 1/2$. Therefore

$$Z = 100\frac{(1+0.04/4)^{4\times 5} - 1}{(1+0.04/4)^{4\times ½} - 1}(1+0.04/4)^{4\times ½} = 100\frac{1.01^{20} - 1}{1.01^2 - 1}1.01^2$$
$$= 100\frac{1.220 - 1}{1.020 - 1}1.020$$
$$= \$1122\pm. \quad \textit{Answer.}$$

Note.—1.01^{20} is the amount of \$1 for 20 years at 1 per cent. compounded annually, and 1.01^2 is the amount of \$1 for two years at 1 per cent. compounded annually. Since Table A does not give amounts for the rate of 1 per cent., the values of 1.01^{20} and 1.01^2 were computed by logarithms (§14).

Table V.—Sinking Funds

In which deposit is made at the *beginning* of each k-year period. (See §29.)

n = life of sinking fund.
k = number of years in one deposit interval
  = or $> 1/h$.
s = amount of \$1 for one year
  $= (1 + i/h)^h$.
i = rate.
h = number of interest periods in one year.

Given	Required	Formula
Deposit, D	Amount, Z	$Z = D\frac{s^n - 1}{s^k - 1}s^k$ (30h)
Deposit, D	Present worth, W	$W = D\frac{s^n - 1}{s^k - 1}\cdot\frac{s^k}{s^n}$ (30i)
Amount, Z	Deposit, D	$D = Z\frac{s^k - 1}{s^n - 1}\cdot\frac{1}{s^k}$ (30j)
Present worth, W	Deposit, D	$D = W\frac{s^k - 1}{s^n - 1}\cdot s^{n-k}$ (30k)
Deposit, D Life, n years Deferred m years	Present worth, W'	$W' = D\frac{s^n - 1}{s^k - 1}\cdot\frac{s^k}{s^{n+m}}$ (30l)
Amount, Z Deposit, D	Life, n	$n = \frac{\log\left[(Z/D)\frac{s^k - 1}{s^k} + 1\right]}{\log s}$ (30m)
Present worth, W Deposit, D	Life, n	$n = \frac{\log\left[\frac{1}{1 - (W/D)\left(1 - \frac{1}{s^k}\right)}\right]}{\log s}$ (30n)
Deposit, D Life, $n = \infty$	Present worth, W	$W = D\frac{s^k}{s^k - 1}$ (30o)
Present worth, W Life, $n = \infty$	Deposit, D	$D = W\frac{s^k - 1}{s^k}$ (30p)
Deposit, D Life, $n = \infty$ Deferred m years	Present worth, W'	$W' = D\frac{s^k}{s^k - 1}\cdot\frac{1}{s^m}$ (30q)

USE OF TABLE VI

The formulas of Table VI assume that the interest period is not longer than the deposit interval; that is, the formulas do not hold when the deposit interval is shorter than the interest period. For example, Eq. 30r will give the amount of the sinking fund in which D is deposited at the end of each half-year if interest is compounded semi-annually, or quarterly, or oftener, but will *not* give the amount if interest is compounded annually.

Example.—If at the *beginning of each* half-year and at the *end of the last* half-year of a five-year period, we deposit \$100 in a savings bank which pays interest at the rate of 4 per cent. compounded quarterly, what will be the accumulated amount at the end of the fifth year?

For this example we use Eq. 30r of Table VI:

$Z = D\dfrac{s^{n+k} - 1}{s^k - 1}$, where $s = (1 + i/h)^h$, $s^{n+k} = (1 + i/h)^{h(n+k)}$ and $s^k = (1 + i/h)^{hk}$. Substituting, we have

$$Z = D\frac{(1 + i/h)^{h(n+k} - 1}{(1 + i/h)^{hk} - 1}.$$

In this example, $D = 100$, $h = 4$, $n = 5$, and $k = 1/2$. Therefore

$$Z = 100\frac{(1 + 0.04/4)^{(5+\frac{1}{2})4} - 1}{(1 + 0.04/4)^{4\times\frac{1}{2}} - 1} = 100\frac{1.01^{22} - 1}{1.01^2 - 1}$$
$$= 100\frac{1.245 - 1}{1.020 - 1}$$
$$= \$1225\pm. \quad \textit{Answer.}$$

Note.—1.01^{22} is the amount of \$1 for 22 years at 1 per cent. compounded annually, and 1.01^2 is the amount of \$1 for two years at 1 per cent compounded annually. Since Table A does not give compound amount for the rate 1 per cent., the values of 1.01^{22} and 1.01^2 were computed by logarithms (§14).

Table VI.—Sinking Funds

In which deposit is made at *beginning of each* and at the *end of the last* k-year period (see §30).

n = life of sinking fund.
k = number of years in one deposit interval
= or > $1/h$.
s = amount of \$1 for one year
= $(1 + i/h)^h$.
i = rate.
h = number of interest periods in one year.

Given	Required	Formulas
Deposit, D	Amount, Z	$Z = D\dfrac{s^{n+k} - 1}{s^k - 1}$ (30r)
Deposit, D	Present worth, W	$W = D\dfrac{s^{n+k} - 1}{s^k - 1} \cdot \dfrac{1}{s^n}$ (30s)
Amount, Z	Deposit, D	$D = Z\dfrac{s^k - 1}{s^{n+k} - 1}$ (30t)
Present worth, W	Deposit, D	$D = W\dfrac{s^k - 1}{s^{n+k} - 1} \cdot s^n$ (30u)
Deposit, D Life, n years Deferred m years	Present worth, W'	$W' = D\dfrac{s^{n+k} - 1}{s^k - 1} \cdot \dfrac{1}{s^{n+m}}$ (30v)
Amount, Z Deposit, D	Life, n	$n = \dfrac{\log\left[(Z/D)\dfrac{s^k - 1}{s^k} + \dfrac{1}{s^k}\right]}{\log s}$ (30w)
Present worth, W Deposit, D	Life, n	$n = \dfrac{\log\left[\dfrac{1}{s^k - (W/D)(s^k - 1)}\right]}{\log s}$ (30x)
Deposit, D Life, $n = \infty$	Present worth, W	$W = D\dfrac{s^k}{s^k - 1}$ (30y)
Present worth, W Life, $n = \infty$	Deposit, D	$D = W\dfrac{s^k - 1}{s^k}$ (30z)
Deposit, D Life, $n = \infty$ Deferred m years	Present worth, W'	$W' = D\dfrac{s^k}{s^k - 1} \cdot \dfrac{1}{s^m}$ (30zz)

USE OF TABLE A

Table A gives the value of s_n of Eq. 18a, Table II, for given values of i, h, and n.

When interest is compounded annually.

Example 1.—What is the amount of \$1 for 15 years at 7 per cent. compounded annually? Under "7 per cent." and opposite "15" of Table A, we find \$2.759, which is the required amount.

Example 2.—What is the amount of \$147.60 for 15 years at 7 per cent. compounded annually? As in the example above, we find the amount of \$1 for the given conditions to be \$2.759; and multiplying this by the principal, we obtain $147.60 \times 2.759 = \$407.22 \pm$. *Answer.*

When interest is compounded semi-annually, or quarterly, etc.

Example 3.—What is the amount of \$1 for 13 years at 5 per cent. compounded semi-annually? Here there are 2 interest periods in the year; therefore $h = 2$, $i/h = 0.05/2 = 0.025$, and $nh = 13 \times 2 = 26$. Opposite "26," under "2-1/2 per cent.," find, in Table A, \$1.900 which is the amount required.

When the number of interest periods is not given in the Table.

Example 4.—What is the amount of \$1 for 82 years at 6 per cent. compounded annually? In Table A, opposite 41 (which is 1/2 of 82) and under 6 per cent. we find 10.90. The amount for 82 years is $10.90^2 = \$118.81 \pm$. *Answer.* (Explanation: $s^n = (s^{n/2})^2 = (s^{n/3})^3 = (s^{n/4})^4$, etc.)

Example 5.—What is the amount of \$1 for 40 years at 8 per cent. compounded quarterly? The number of interest periods is $hn = 4 \times 40 = 160$, which is beyond the reach of the Table. $i/h = 0.08/4 = 0.02$. In Table A, opposite "40" and under "2 per cent.," find 2.208. The required amount is

$$2.208^4 = \$22.69. \quad \textit{Answer.}$$

Table A.—Compound Amount (s_n) of $1

For n years when interest is compounded annually at

n	2%	2-½%	3%	3-½%	4%	4-½%	5%	6%	7%	8%
1	1.020	1.025	1.030	1 035	1.040	1 045	1 050	1 060	1.070	1 080
2	1 040	1 051	1 061	1.071	1.082	1.092	1 103	1.124	1 145	1.166
3	1 061	1 077	1 092	1.109	1.125	1.141	1 158	1 191	1 225	1 260
4	1.082	1.104	1.126	1.148	1.170	1.193	1.216	1 262	1 311	1.360
5	1 104	1.131	1.159	1.188	1.217	1.246	1.276	1 338	1.403	1.469
6	1.126	1.160	1.194	1 229	1 265	1 302	1 340	1 419	1 501	1 587
7	1.149	1.189	1 230	1.272	1.316	1.361	1.407	1 504	1 606	1.714
8	1 172	1 218	1 267	1 317	1 369	1.422	1.477	1 594	1.718	1.851
9	1 195	1.249	1 305	1 363	1.423	1 486	1 551	1.689	1 838	1.999
10	1 219	1.280	1 344	1 411	1.480	1.553	1 629	1 791	1.967	2 159
11	1.243	1.312	1 384	1 460	1 539	1.623	1 710	1.898	2 105	2.332
12	1.268	1.345	1 426	1 511	1 601	1 696	1.796	2.012	2 252	2.518
13	1.294	1.379	1 469	1 564	1.665	1.772	1 886	2 133	2 410	2.720
14	1 319	1.413	1.513	1 619	1.732	1 852	1 980	2.261	2 579	2.937
15	1 346	1.448	1 558	1 675	1.801	1.935	2.079	2 397	2 759	3.172
16	1 373	1.485	1 605	1.734	1.873	2.022	2 183	2.540	2 952	3.426
17	1.400	1.522	1 653	1.795	1.948	2 113	2 292	2 693	3 159	3.700
18	1 428	1 560	1.702	1.857	2.026	2 208	2 407	2 854	3 380	3.996
19	1.457	1 599	1 754	1 923	2.107	2.308	2 527	3 026	3 617	4.316
20	1 486	1 639	1 806	1 990	2 191	2 412	2.653	3 207	3 870	4.661
21	1.516	1.680	1 860	2 059	2 279	2.520	2.786	3 400	4 141	5.034
22	1 546	1.722	1 916	2 132	2 370	2 634	2.925	3 604	4 430	5.437
23	1.577	1 765	1.974	2 206	2 465	2.752	3.072	3 820	4 741	5 871
24	1.608	1.809	2.033	2 283	2.563	2.876	3 225	4.049	5.072	6.341
25	1.641	1 854	2 094	2 363	2 666	3 005	3.386	4.292	5 427	6.848
26	1.673	1 900	2.157	2.446	2.772	3.141	3.556	4.549	5 807	7 396
27	1 707	1.948	2 221	2.532	2 883	3 282	3 733	4 822	6.214	7.988
28	1 741	1 996	2 288	2 620	2 999	3 430	3 920	5.112	6.649	8 627
29	1 776	2.046	2.357	2.712	3 119	3.584	4.116	5.418	7 114	9 317
30	1 811	2 098	2.427	2 807	3.243	3.745	4 322	5 743	7.612	10.06
31	1 848	2.150	2.500	2.905	3.373	3.914	4.538	6.088	8 145	10 87
32	1.885	2 204	2.575	3.007	3 508	4.090	4.765	6.453	8.715	11.74
33	1 922	2.259	2.652	3.112	3.648	4.274	5 003	6.841	9 325	12 68
34	1.961	2.315	2.732	3.221	3.794	4.466	5.253	7.251	9 978	13 69
35	2 000	2.373	2.814	3.334	3.946	4.667	5.516	7 686	10.68	14 79
36	2.040	2.433	2.898	3.450	4.104	4 877	5 792	8.147	11.42	15 97
37	2.081	2.493	2.985	3.571	4.268	5.097	6 081	8 636	12.22	17.25
38	2.122	2 556	3.075	3.696	4 439	5 326	6 385	9.154	13 08	18 63
39	2.165	2 620	3.167	3.825	4.616	5.566	6 705	9.704	13.99	20.12
40	2 208	2.685	3 262	3.959	4.801	5 816	7.040	10 29	14 97	21.72
41	2 252	2.752	3.360	4.098	4.993	6.078	7.392	10 90	16.02	23.46
42	2 297	2.821	3.461	4.241	5.193	6.352	7.762	11 56	17.14	25 34
43	2 343	2.892	3 565	4 390	5.400	6.637	8.150	12.25	18.34	27 37
44	2.390	2.964	3 671	4.543	5 617	6 936	8.557	12.99	19 63	29 56
45	2 438	3 038	3 782	4 702	5.841	7.248	8.985	13 76	21.00	31 92
46	2 487	3.114	3 895	4 867	6.075	7.574	9 434	14.59	22.47	34 47
47	2 536	3.192	4.012	5 037	6 318	7.915	9.906	15 47	24 05	37.23
48	2 587	3.271	4 132	5.214	6.571	8 271	10.40	16 39	25.73	40 21
49	2.639	3.353	4 256	5.396	6.833	8 644	10.92	17.38	27.53	43.43
50	2 692	3.437	4.384	5 585	7.107	9 033	11 47	18 42	29 46	46.90
55	2.972	3.889	5.082	6.633	8.646	11 26	14 64	24 65	41.32	68 91
60	3.281	4.400	5.892	7.878	10.52	14 03	18.68	32 99	57.95	101.3
65	3 623	4 978	6.830	9 357	12 80	17.48	23 84	44 14	81.27	148.8
70	4.000	5.632	7 918	11.11	15.57	21.78	30.43	59 08	114.0	218.6
75	4.416	6.372	9.179	13 20	18.95	27 15	38 83	79 06	159.9	321.2
80	4.875	7.210	10 64	15.68	23.05	33 83	49.56	105.8	224.2	472.0
85	5.383	8.157	12.34	18.62	28.04	42 16	63.25	141 6	314.5	693.5
90	5 943	9.229	14 30	22 11	34.12	52 54	80.73	189.5	441.1	1019.0
95	6 562	10.44	16 58	26.26	41.51	65.47	103 0	253.5	618.7	1497.0
100	7.245	11 81	19 22	31 19	50 50	81 59	131 5	339 3	867.7	2200

USE OF TABLE B

Table B gives the value of p_n of Eq. 20b, Table II, for given values of i, h, and n.

When interest is compounded annually.

Example 1.–What is the present worth of \$1 due 17 years hence, interest being at 5 per cent. compounded annually? Opposite "17" and under "5 per cent." of Table B, we find \$0.4363, which is the required present worth.

Example 2.—What is the present worth of \$100 under the conditions above? For the given conditions we find, as in Example 1, the present worth of \$1 to be \$0.4363; and multiply this by 100, obtaining the required present worth, \$43.63.

When interest is compounded semi-annually, or quarterly, etc.

Example 3.—What is the present worth of \$1 due 10 years hence, interest being at 8 per cent. compounded quarterly? Here $h = 4$, and $n = 10$; hence the number of interest periods is $hn = 40$. $i/h = 0.08/4 = 0.02$. Opposite "40" and under "2 per cent." of Table B, we find \$0.4529, which is the required present worth.

When the number of interest periods is not given in the table.

Example 4.—What is the present worth of \$1 due 60 years hence, with interest at 4 per cent. compounded semi-annually? Here the number of interest periods is $hn = 2 \times 60 = 120$, which is not in the table. 60, which is one-half of 120, is in the table. $i/h = 0.04/2 = 0.02$. Opposite "60" and under "2 per cent.," in Table B, we find 0.3048. The required present worth is $0.3048^2 = \$0.0929+$. (Explanation: $p_n = p^n = (p^{n/2})^2 = (p^{n/3})^3$, and so on.)

Table B.—Present Worth (p_n) of $1

Due n years hence, when interest is compounded annually at

n	2%	2-½%	3%	3-½%	4%	4-½%	5%	6%	7%	8%
1	0 9804	0.9756	0 9709	0 9662	0.9615	0 9569	0 9524	0 9434	0 9346	0 9259
2	0 9612	0.9518	0 9426	0 9335	0.9246	0 9157	0 9070	0.8900	0 8734	0 8573
3	0 9423	0 9286	0.9151	0.9019	0.8890	0 8763	0 8638	0 8396	0 8163	0 7938
4	0 9238	0.9060	0 8885	0.8714	0.8548	0 8386	0.8227	0.7921	0.7629	0 7350
5	0 9057	0 8839	0.8626	0.8420	0.8219	0 8025	0 7835	0.7473	0 7130	0.6806
6	0.8880	0 8623	0.8375	0.8135	0.7903	0 7679	0 7462	0.7050	0.6663	0 6302
7	0 8706	0 8413	0.8131	0.7860	0.7599	0 7348	0.7107	0.6651	0 6227	0.5835
8	0.8535	0.8207	0.7894	0.7594	0.7307	0.7032	0 6768	0.6274	0 5820	0 5403
9	0.8368	0 8007	0.7664	0 7337	0.7026	0 6729	0 6446	0.5919	0.5439	0 5002
10	0 8203	0.7812	0 7441	0 7089	0.6756	0 6439	0 6139	0.5584	0 5083	0.4632
11	0 8043	0 7621	0.7224	0 6849	0.6496	0.6162	0 5847	0.5268	0 4751	0.4289
12	0.7885	0 7436	0 7014	0.6618	0.6246	0.5897	0 5568	0.4970	0.4440	0.3971
13	0 7730	0.7254	0.6810	0.6394	0.6006	0.5643	0.5303	0.4688	0.4150	0.3677
14	0.7579	0.7077	0.6611	0.6178	0.5775	0.5400	0.5051	0 4423	0 3878	0.3405
15	0 7430	0.6905	0.6419	0.5969	0.5553	0.5167	0.4810	0 4173	0 3624	0 3152
16	0 7284	0.6736	0.6232	0 5767	0.5339	0.4945	0 4581	0 3936	0 3387	0 2919
17	0 7142	0.6572	0 6050	0.5572	0.5134	0 4732	0.4363	0 3714	0 3166	0.2703
18	0.7002	0 6412	0.5874	0 5384	0.4936	0.4528	0.4155	0 3503	0 2959	0.2502
19	0 6864	0.6255	0 5703	0 5202	0.4746	0 4333	0 3957	0.3305	0.2765	0 2317
20	0.6730	0 6103	0.5537	0 5026	0 4564	0 4146	0.3769	0.3118	0 2584	0 2145
21	0 6598	0 5954	0 5375	0.4856	0.4388	0 3968	0 3589	0 2942	0.2415	0.1987
22	0 6468	0 5809	0 5219	0.4692	0.4220	0.3797	0 3418	0 2775	0 2257	0 1839
23	0 6342	0 5667	0.5067	0.4533	0.4057	0.3634	0 3256	0 2618	0 2109	0 1703
24	0 6217	0.5529	0.4919	0.4380	0.3901	0.3477	0.3101	0 2470	0.1971	0 1577
25	0 6095	0 5394	0 4776	0 4231	0.3751	0.3327	0 2953	0 2330	0 1842	0 1460
26	0 5976	0 5262	0 4637	0.4088	0.3607	0.3184	0 2812	0 2198	0 1722	0 1352
27	0.5859	0 5134	0.4502	0.3950	0.3468	0.3047	0.2678	0 2074	0.1609	0.1252
28	0 5744	0.5009	0 4371	0.3817	0.3335	0 2916	0 2551	0 1956	0.1504	0.1159
29	0 5631	0 4887	0 4243	0.3687	0.3207	0.2790	0 2429	0.1846	0 1406	0.1073
30	0.5521	0 4767	0.4120	0 3563	0.3083	0 2670	0.2314	0 1741	0 1314	0.0994
31	0 5412	0.4651	0.4000	0.3442	0.2965	0.2555	0.2204	0.1643	0.1228	0.0920
32	0.5306	0.4538	0.3883	0 3326	0.2851	0.2445	0 2099	0.1550	0.1147	0.0852
33	0 5202	0.4427	0.3770	0.3213	0.2741	0.2340	0.1999	0 1462	0.1072	0 0789
34	0 5100	0.4319	0.3660	0.3105	0.2636	0 2239	0.1904	0 1379	0.1002	0 0730
35	0 5000	0 4214	0.3554	0.3000	0.2534	0.2143	0.1813	0 1301	0.0937	0 0676
36	0 4902	0 4111	0.3450	0.2898	0.2437	0 2050	0.1727	0 1227	0.0875	0.0626
37	0.4806	0.4011	0.3350	0 2800	0.2343	0.1962	0.1644	0 1158	0.0818	0 0580
38	0 4712	0 3913	0.3252	0.2706	0.2253	0.1878	0.1566	0.1092	0 0765	0 0537
39	0.4619	0.3817	0.3158	0 2614	0.2166	0.1797	0.1491	0 1031	0 0715	0 0497
40	0.4529	0 3724	0.3066	0.2526	0.2083	0 1719	0 1420	0 0972	0.0668	0 0460
41	0.4440	0.3633	0.2976	0.2440	0.2003	0.1645	0.1353	0 0917	0 0624	0 0426
42	0 4353	0.3545	0.2890	0.2358	0.1926	0.1574	0 1288	0 0865	0 0582	0 0395
43	0 4268	0.3458	0.2805	0.2278	0.1852	0.1507	0.1227	0 0816	0 0545	0 0365
44	0 4184	0.3374	0.2724	0.2201	0.1780	0.1442	0 1169	0.0770	0 0509	0 0338
45	0.4102	0.3292	0.2644	0.2127	0.1712	0.1380	0 1113	0 0727	0 0476	0 0313
46	0.4022	0.3211	0.2567	0.2055	0.1646	0.1320	0.1060	0.0685	0 0445	0.0290
47	0.3943	0.3133	0.2493	0.1985	0.1583	0.1263	0 1009	0.0647	0 0416	0.0269
48	0.3865	0.3057	0.2420	0.1918	0.1522	0.1209	0.0961	0.0610	0 0389	0.0249
49	0.3790	0.2982	0.2350	0 1853	0.1463	0 1157	0.0916	0.0575	0.0363	0.0230
50	0 3715	0.2909	0 2281	0 1791	0.1407	0.1107	0 0872	0 0543	0 0339	0 0213
55	0 3365	0.2572	0 1968	0.1508	0.1157	0 0888	0 0683	0 0406	0.0242	0 0145
60	0 3048	0.2273	0 1697	0.1269	0.0951	0.0713	0.0535	0 0303	0 0173	0 0099
65	0 2761	0.2009	0.1464	0.1069	0.0781	0 0572	0.0419	0.0227	0 0123	0 0067
70	0.2500	0.1776	0.1263	0.0900	0.0642	0 0459	0.0329	0.0169	0 0088	0 0046
75	0 2265	0.1569	0.1089	0.0758	0.0528	0.0368	0 0258	0 0127	0.0063	0 0031
80	0 2051	0.1387	0.0940	0.0638	0.0434	0 0296	0.0202	0 0095	0 0045	0.0021
85	0 1858	0.1226	0.0811	0.0537	0.0357	0.0237	0.0158	0.0071	0.0032	0 0014
90	0.1683	0.1084	0.0699	0.0452	0.0293	0.0190	0 0124	0 0053	0 0023	0 0010
95	0 1524	0.0958	0.0603	0 0381	0.0241	0 0153	0.0097	0.0039	0 0016	0.0007
100	0.1380	0.0846	0 0520	0.0321	0.0198	0 0123	0 0076	0 0029	0 0012	0 0005

USE OF TABLE C

Table C gives the value of z_n of Eq. 31, Table III, for given values of i, h, and n.

When the deposit is made at the end of each year and interest is compounded annually.

Example 1.—At the end of each year for 20 years we deposit $1 in a savings bank which pays 4 per cent. compounded annually. What is the amount of the sinking fund thus created? In Table C, opposite "20" and under "4 per cent.," we find $29.78, which is the required amount.

Example 2.—If the deposit were $100, instead of $1, in the foregoing example, the required amount would be 100 × 29.78 = $2978±.

When the deposit is made at the end of each half-year and interest is compounded semi-annually.

Example 3.—If we deposit $1 at the end of each half-year for 10 years, and interest on deposits is 8 per cent. compounded semi-annually, what is the amount of the series of deposits? $i/h = 0.08/2 = 0.04$; and $hn = 2 \times 10 = 20$. In Table C, opposite "20" and under "4 per cent.," we find $29.78, which is the required amount.

When the deposit is made at the end of each interest period, whether the interest period be one-half year, or one-fourth year, or one-sixth year, find the required amount in Table C, opposite hn and under $100i/h$ per cent.

Table C.—Amount (z_n) of Sinking Fund

Of which the life is n years, the end-of-year deposits are $1 each, and the interest is compounded annually at

n	2%	2-½%	3%	3-½%	4%	4-½%	5%	6%	7%	8%
1	1 000	1 000	1 000	1 000	1.000	1 000	1 000	1.000	1 000	1.000
2	2 020	2 025	2 030	2 035	2.040	2.045	2.050	2 060	2 070	2 080
3	3.060	3 076	3 091	3.106	3.122	3.137	3 153	3 184	3 215	3 246
4	4 122	4 153	4.184	4 215	4.246	4.278	4 310	4.375	4 440	4 506
5	5 204	5 256	5.309	5 362	5.416	5.471	5 526	5 637	5 751	5 867
6	6 308	6 388	6 468	6 550	6.633	6 717	6.802	6 975	7.153	7.336
7	7 434	7.547	7 662	7.779	7.898	8.019	8 142	8 394	8 654	8 923
8	8 583	8 736	8 892	9 052	9.214	9 380	9 549	9 897	10 26	10.64
9	9 755	9 955	10.16	10 37	10 58	10 80	11.03	11.49	11.98	12.49
10	10 95	11 20	11 46	11.73	12.01	12 29	12 58	13 18	13 82	14.49
11	12 17	12 48	12 81	13.14	13 49	13.84	14 21	14.97	15 78	16 65
12	13 41	13 80	14.19	14.60	15 03	15 46	15 92	16 87	17 89	18 98
13	14 68	15 14	15.62	16 11	16 63	17 16	17.71	18 88	20.14	21 50
14	15.97	16.52	17.09	17.68	18 29	18.93	19 60	21 02	22 55	24 21
15	17.29	17 93	18 60	19 30	20 02	20 78	21.58	23.28	25 13	27 15
16	18.64	19.38	20.16	20.97	21.82	22.72	23 66	25 67	27 89	30 32
17	20.01	20 86	21 76	22.71	23.70	24 74	25 84	28 21	30 84	33 75
18	21 41	22 39	23 41	24.50	25 65	26 86	28.13	30 91	34 00	37 45
19	22.84	23.95	25.12	26.36	27 67	29 06	30 54	33 76	37 38	41 45
20	24 30	25.54	26 87	28.28	29 78	31 37	33 07	36 79	41.00	45 76
21	25 78	27.18	28.68	30 27	31 97	33 78	35 72	39 99	44 87	50.42
22	27.30	28.86	30.54	32.33	34 25	36 30	38.51	43 39	49 01	55 46
23	28.84	30 58	32 45	34 46	36 62	38 94	41 43	47 00	53 44	60 89
24	30.42	32.35	34 43	36 67	39 08	41 69	44 50	50 82	58 18	66 76
25	32 03	34.16	36 46	38.95	41 65	44.57	47 73	54 86	63 25	73.11
26	33.67	36.01	38 55	41.31	44 31	47 57	51.11	59 16	68.68	79 95
27	35.34	37.91	40.71	43 76	47 08	50.71	54.67	63.71	74.48	87 35
28	37.05	39.86	42.93	46 29	49 97	53.99	58 40	68 53	80.70	95 34
29	38.79	41.86	45.22	48 91	52 97	57.42	62.32	73 64	87.35	104.0
30	40.57	43.90	47.58	51.62	56.08	61 01	66 44	79 06	94.46	113 3
31	42.38	46.00	50.00	54.43	59.33	64 75	70 76	84 80	102 1	123 3
32	44 23	48.15	52 50	57.33	62 70	68 67	75 30	90 89	110 2	124 2
33	46 11	50.35	55.08	60 34	66 21	72 76	80 06	97 34	118 9	146 0
34	48.03	52.61	57.73	63 45	69.86	77.03	85 07	104 2	128.3	158.6
35	49 99	54.93	60.46	66.67	73.65	81.50	90 32	111.4	138 2	172 3
36	51.99	57.30	63.28	70 01	77.60	86.16	95.84	119.1	148 9	187 1
37	54 03	59.73	66.17	73 46	81 70	91.04	101 6	127 3	160 3	203.1
38	56.11	62.23	69.16	77.03	85.97	96 14	107 7	135 9	172 6	220 3
39	58 24	64.78	72.23	80.72	90 41	101 5	114.1	145.1	185 6	238.9
40	60 40	67.40	75 40	84.55	95 03	107 0	120 8	154.8	199.6	259.1
41	62 61	70 09	78 66	88 51	99 83	112 8	127 8	165 0	214 6	280.8
42	64.86	72 84	82.02	92 61	104 8	118 9	135.2	176.0	230 6	304 2
43	67 16	75 66	85 48	96 85	110 0	125 3	143 0	187.5	247 8	329.6
44	69 50	78.55	89 05	101.2	115.4	131 9	151 1	199 8	266 1	356.9
45	71.89	81.52	92 72	105.8	121.0	138 8	159 7	212 7	285 7	386 5
46	74 33	84.55	96 50	110.5	126 9	146 1	168 7	226.5	306 8	418 4
47	76 82	87.67	100 4	115 4	132 9	153 7	178 1	241 1	329 2	452 9
48	79.35	90 86	104.4	120.4	139.3	161.6	188 0	256.6	353 3	490 1
49	81.94	94.13	108 5	125 6	145.8	169 9	198 4	273.0	379 0	530 3
50	84 58	97 48	112 8	131 0	152 7	178 5	209.3	290.3	406.5	573.8
55	98.59	115 6	136.1	160 9	191 2	227 9	272.7	394.2	575 9	848 9
60	114.1	136 0	163.1	196 5	238 0	289.5	353.6	533.1	813 5	1253.
65	131.1	159.1	194.3	238.8	295 0	366 2	456.8	719.1	1147.	1847.
70	150 0	185 3	230 6	288 9	364 3	461 9	588 5	967.9	1614.	2720.
75	170 8	214.9	272 6	348 5	448 6	581.0	756.7	1301	2270.	4003.
80	193 8	248 4	321.4	419.3	551 2	729 6	971.2	1747.	3189	5887.
85	219.1	286 3	377.9	503.4	676.1	914 6	1245	2343.	4479.	8656.
90	247.2	329 2	443.3	603 2	828.	1145	1595.	3141.	6287.	12724.
95	278.1	377 7	519.3	721.8	1013.	1433.	2041.	4209.	8824.	18702
100	312.2	432 5	607 3	862.6	1238.	1791.	2610.	5638.	12382.	27485.

USE OF TABLE D

Table D gives values of w_n of Eq. 32, Table III, for given values of i, h, and n.

When the payment is made (or avoided) at the end of each year, and interest is compounded annually.

Example 1.—What can we afford to pay now in order to effect a saving of $1 at the end of each year for seven years, if money is worth 8 per cent. compounded annually? In Table D, opposite "7" and under "8 per cent.," we find $5.206, which is the required present sum.

Example 2.—What is the capitalized value of a yearly expense of $100 for 25 years, if interest is 6 per cent. compounded annually? Opposite "25" and under "6 per cent.," Table D, we find $12.78, which multiplied by 100 gives $1278±, which is the capitalized value sought.

When the payment is made (or avoided) at the end of each interest period which is less than one year.

Example 3.—What principal is required to yield an annuity of $200, one-half of which is to be paid at the end of each half-year, for 14 years, if interest is 6 per cent. compounded semi-annually? $i/h = 0.06/2 = 0.03$, $hn = 2 \times 14 = 28$. Opposite "28" and under "3 per cent.," Table D, we find $18.76; and this multiplied by 100 gives $1876±, which is the required principal.

Table D.—Principal (w_n) which will Produce Annuity of $1

At the end of each year for n years, when the interest is compounded annually at

n	2%	2-½%	3%	3-½%	4%	4-½%	5%	6%	7%	8%
1	0 9804	0.9756	0 9709	0 9662	0 9615	0 9569	0 9524	0 9434	0.9346	0.9259
2	1.942	1 927	1 913	1 900	1 886	1 873	1 859	1 833	1 808	1 783
3	2 884	2.856	2 829	2 802	2 775	2.749	2 723	2 673	2 624	2 577
4	3 808	3 762	3.717	3.673	3.630	3.588	3 546	3.465	3.387	3 312
5	4 713	4 646	4 580	4.515	4.452	4.390	4 329	4.212	4.100	3 993
6	5 601	5 508	5 417	5 329	5 242	5 158	5 076	4 917	4 767	4 623
7	6 472	6 349	6 230	6 115	6 002	5 893	5.786	5 582	5 389	5 206
8	7 325	7 170	7 020	6 874	6 733	6 596	6 463	6 210	5 971	5 747
9	8 162	7.971	7 786	7 608	7 435	7 269	7 108	6 802	6 515	6 247
10	8 983	8 752	8 530	8 317	8 111	7 913	7 722	7 360	7 024	6 710
11	9 787	9 514	9 253	9 002	8 760	8 529	8.306	7 887	7 499	7.139
12	10 58	10.26	9 954	9 663	9.385	9.119	8 863	8.384	7 943	7.536
13	11 35	10 98	10 63	10 30	9.986	9 683	9 394	8 853	8 358	7.904
14	12.11	11 69	11.30	10 92	10 56	10 22	9 899	9.295	8.745	8.244
15	12.85	12 38	11.94	11 52	11.12	10.74	10 38	9 712	9.108	8.559
16	13 58	13.06	12 56	12 09	11.65	11 23	10 84	10 11	9.447	8 851
17	14.29	13 71	13 17	12.65	12 17	11 71	11 27	10 48	9 763	9.122
18	14 99	14.35	13 75	13 19	12.66	12 16	11 69	10 83	10 06	9.372
19	15.68	14 98	14 32	13 71	13 13	12 59	12 09	11.16	10 24	9.604
20	16 35	15.59	14.88	14 21	13.59	13.01	12 46	11 47	10 59	9 818
21	17 01	16 18	15 42	14.70	14 03	13 40	12 82	11 76	10.84	10 02
22	17.66	16.77	15 94	15 17	14 45	13 78	13.16	12 04	11 06	10 20
23	18.29	17 33	16 44	15 62	14 86	14 15	13 49	12 30	11.27	10 37
24	18.91	17.88	16.94	16 06	15 25	14 50	13 80	12 55	11.47	10 53
25	19.52	18 42	17.41	16 48	15 62	14 83	14 09	12 78	11 65	10 67
26	20 12	18 95	17 88	16 89	15 98	15 15	14 38	13 00	11 83	10 81
27	20 71	19 46	18.33	17 29	16 33	15 45	14 64	13 21	11 99	10 94
28	21 28	19 96	18 76	17 67	16 66	15 74	14.90	13 41	12 14	11 05
29	21.84	20 45	19 19	18 04	16 98	16 02	15.14	13.59	12 28	11 16
30	22.40	20 93	19.60	18 39	17 29	16.29	15.37	13 76	12.41	11 26
31	22 94	21 40	20 00	18 74	17 59	16 54	15 59	13 93	12 53	11 35
32	23 47	21 85	20 39	19 07	17.87	16 79	15 80	14 08	12.65	11 43
33	23 99	22 29	20 77	19 39	18.15	17.02	16 00	14.23	12 75	11 51
34	24 50	22 72	21.13	19 70	18.41	17 25	16.19	14 37	12 85	11.59
35	25.00	23 15	21 49	20.00	18.66	17 46	16 37	14 50	12 95	11 65
36	25 49	23 56	21 83	20 29	18.91	17 67	16 55	14 62	13 04	11 72
37	25 97	23 96	22 17	20 57	19.14	17 86	16 71	14 74	13.12	11 78
38	26 44	24 35	22 49	20 84	19.37	18 05	16 87	14 85	13 19	11 83
39	26 90	24 73	22 81	21.10	19.58	18 23	17 02	14.95	13 26	11 88
40	27 36	25 10	23.11	21 36	19.79	18 40	17 16	15.05	13 33	11 92
41	27 80	25 47	23 41	21 60	19 99	18 57	17 29	15 14	13 39	11 97
42	28 23	25.82	23 70	21 83	20.19	18 72	17 42	15.22	13.45	12 01
43	28 66	26 17	23 98	22 06	20.37	18 87	17.55	15 31	13 51	12 04
44	29 08	26 50	24.25	22 28	20.55	19 02	17 66	15 38	13 56	12 08
45	29 49	26 83	24.52	22 50	20.72	19 16	17 77	15 46	13 61	12.11
46	29 89	27 15	24.78	22 70	20.88	19.29	17 88	15.52	13.65	12 14
47	30 29	27 47	25 02	22 90	21.04	19.41	17 98	15 59	13 69	12 16
48	30 67	27 77	25.27	23 09	21.20	19.54	18.08	15 65	13.73	12.19
49	31 05	28 07	25 50	23 28	21.34	19 65	18 17	15.71	13.77	12 21
50	31 42	28 36	25 73	23 46	21.48	19 76	18 26	15 76	13 80	12 23
55	33.17	29 71	26 77	24 26	22 11	20.25	18 63	15.99	13 94	12 32
60	34 76	30 91	27 68	24 94	22.62	20.64	18 93	16.16	14 04	12 38
65	36.20	31 96	28 45	25 52	23 05	20.95	19 16	16 29	14 11	12.42
70	37 50	32.90	29.12	26 00	23 39	21.20	19 34	16 38	14 16	12 44
75	38.68	33 72	29 70	26 41	23 68	21 40	19.48	16 46	14 20	12.46
80	39 74	34 45	30 20	26.75	23 92	21.57	19 60	16.51	14 22	12 47
85	40 71	35 10	30 63	27.04	24 11	21 70	19 68	16 55	14 24	12.48
90	41.59	35.67	31.00	27.28	24.27	21 80	19 75	16 58	14 25	12 49
95	42 38	36 17	31 32	27.48	24.40	21 88	19.81	16 60	14.26	12.49
100	43.10	36.61	31 60	27.66	24.50	21 95	19 85	16.62	14 27	12.49

USE OF TABLE E

Table E gives the value of d_n of Eq. 33, Table III, for given values of i, h, and n.

When the deposit is made at the end of each year and interest is compounded annually.

Example 1.—What sum must be deposited at the end of each year in a savings bank paying 4 per cent. compounded annually, in order that the accumulation at the end of 18 years may be $1? Opposite "18" and under "4 per cent.," Table E, we find $0.0390, which is the required deposit.

Example 2.—What end-of-year deposit is required to redeem $2000 at the end of seven years, if interest on deposits is 5 per cent. compounded annually? Opposite "7" and under "5 per cent.," Table E, we find $0.1228, which multiplied by 2000 gives $245.60±, which is the deposit required.

When the deposit is made at the end of each interest period which is less than one year.

Example 3.—What deposit is required at the end of each half-year to redeem $1 at the end of 16 years, if interest on deposits is 4 per cent. compounded semi-annually? $i/h = 0.04/2 = 0.02$. $hn = 2 \times 16 = 32$. Opposite "32" and under "2 per cent.," Table E, we find $0.0226, which is the required deposit.

Table E.—Deposit (d_n) in Sinking Fund

The end-of-year deposit which will cause the sinking fund to amount to $1 in n years when the interest is compounded annually at

n	2%	2-½%	3%	3-½%	4%	4-½%	5%	6%	7%	8%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.4950	0.4938	0.4926	0.4914	0.4902	0.4890	0.4878	0.4854	0.4831	0.4808
3	0.3268	0.3251	0.3235	0.3219	0.3203	0.3188	0.3172	0.3141	0.3111	0.3080
4	0.2426	0.2408	0.2390	0.2373	0.2355	0.2337	0.2320	0.2286	0.2252	0.2219
5	0.1922	0.1902	0.1884	0.1865	0.1846	0.1828	0.1810	0.1774	0.1739	0.1705
6	0.1585	0.1565	0.1546	0.1527	0.1508	0.1489	0.1470	0.1434	0.1398	0.1363
7	0.1345	0.1325	0.1305	0.1285	0.1266	0.1247	0.1228	0.1191	0.1156	0.1121
8	0.1165	0.1145	0.1125	0.1105	0.1085	0.1066	0.1047	0.1010	0.0975	0.0940
9	0.1025	0.1005	0.0984	0.0964	0.0945	0.0926	0.0907	0.0870	0.0835	0.0801
10	0.0913	0.0893	0.0872	0.0852	0.0833	0.0814	0.0795	0.0759	0.0724	0.0690
11	0.0822	0.0801	0.0781	0.0761	0.0741	0.0722	0.0704	0.0668	0.0634	0.0601
12	0.0746	0.0725	0.0705	0.0685	0.0666	0.0647	0.0628	0.0593	0.0559	0.0527
13	0.0681	0.0660	0.0640	0.0621	0.0601	0.0583	0.0565	0.0530	0.0497	0.0465
14	0.0626	0.0605	0.0585	0.0566	0.0547	0.0528	0.0510	0.0476	0.0443	0.0413
15	0.0578	0.0558	0.0538	0.0518	0.0499	0.0481	0.0463	0.0430	0.0398	0.0368
16	0.0537	0.0516	0.0496	0.0477	0.0458	0.0440	0.0423	0.0390	0.0359	0.0330
17	0.0500	0.0479	0.0460	0.0440	0.0422	0.0404	0.0387	0.0354	0.0324	0.0296
18	0.0467	0.0447	0.0427	0.0408	0.0390	0.0372	0.0355	0.0324	0.0294	0.0267
19	0.0438	0.0418	0.0398	0.0379	0.0361	0.0344	0.0327	0.0296	0.0268	0.0241
20	0.0412	0.0391	0.0372	0.0354	0.0336	0.0319	0.0302	0.0272	0.0244	0.0219
21	0.0388	0.0368	0.0349	0.0330	0.0313	0.0296	0.0280	0.0250	0.0223	0.0198
22	0.0366	0.0346	0.0327	0.0309	0.0292	0.0275	0.0260	0.0230	0.0204	0.0180
23	0.0347	0.0327	0.0308	0.0290	0.0273	0.0257	0.0241	0.0213	0.0187	0.0164
24	0.0329	0.0309	0.0290	0.0273	0.0256	0.0240	0.0225	0.0197	0.0172	0.0150
25	0.0312	0.0293	0.0274	0.0257	0.0240	0.0224	0.0210	0.0182	0.0158	0.0137
26	0.0297	0.0278	0.0259	0.0242	0.0226	0.0210	0.0196	0.0169	0.0146	0.0125
27	0.0283	0.0264	0.0246	0.0229	0.0212	0.0197	0.0183	0.0157	0.0134	0.0114
28	0.0270	0.0251	0.0233	0.0216	0.0200	0.0185	0.0171	0.0146	0.0124	0.0105
29	0.0258	0.0239	0.0221	0.0204	0.0189	0.0174	0.0160	0.0136	0.0114	0.0096
30	0.0246	0.0228	0.0210	0.0194	0.0178	0.0164	0.0151	0.0126	0.0106	0.0088
31	0.0236	0.0217	0.0200	0.0184	0.0169	0.0154	0.0141	0.0118	0.0098	0.0081
32	0.0226	0.0208	0.0190	0.0174	0.0159	0.0146	0.0133	0.0110	0.0091	0.0075
33	0.0217	0.0199	0.0182	0.0166	0.0151	0.0137	0.0125	0.0103	0.0084	0.0069
34	0.0208	0.0190	0.0173	0.0158	0.0143	0.0130	0.0118	0.0096	0.0078	0.0063
35	0.0200	0.0182	0.0165	0.0150	0.0136	0.0123	0.0111	0.0090	0.0072	0.0058
36	0.0192	0.0175	0.0158	0.0143	0.0129	0.0116	0.0104	0.0084	0.0067	0.0053
37	0.0185	0.0167	0.0151	0.0136	0.0122	0.0110	0.0098	0.0079	0.0062	0.0049
38	0.0178	0.0161	0.0145	0.0130	0.0116	0.0104	0.0093	0.0074	0.0058	0.0045
39	0.0172	0.0154	0.0138	0.0124	0.0111	0.0099	0.0088	0.0069	0.0054	0.0042
40	0.0166	0.0148	0.0133	0.0118	0.0105	0.0093	0.0083	0.0065	0.0050	0.0039
41	0.0160	0.0143	0.0127	0.0113	0.0100	0.0089	0.0078	0.0061	0.0047	0.0036
42	0.0154	0.0137	0.0122	0.0108	0.0095	0.0084	0.0074	0.0057	0.0043	0.0033
43	0.0149	0.0132	0.0117	0.0103	0.0091	0.0080	0.0070	0.0053	0.0040	0.0030
44	0.0144	0.0427	0.0112	0.0099	0.0087	0.0076	0.0066	0.0050	0.0038	0.0028
45	0.0139	0.0123	0.0108	0.0095	0.0083	0.0072	0.0063	0.0047	0.0035	0.0026
46	0.0135	0.0118	0.0104	0.0091	0.0079	0.0068	0.0059	0.0044	0.0033	0.0024
47	0.0130	0.0114	0.0100	0.0087	0.0075	0.0065	0.0056	0.0041	0.0030	0.0022
48	0.0126	0.0110	0.0096	0.0083	0.0072	0.0062	0.0053	0.0039	0.0028	0.0020
49	0.0122	0.0106	0.0092	0.0080	0.0069	0.0059	0.0050	0.0037	0.0026	0.0019
50	0.0118	0.0103	0.0089	0.0076	0.0066	0.0056	0.0048	0.0034	0.0025	0.0017
55	0.0101	0.0087	0.0073	0.0062	0.0052	0.0044	0.0037	0.0025	0.0017	0.0012
60	0.0088	0.0074	0.0061	0.0051	0.0042	3.0035	0.0028	0.0019	0.0012	0.0008
65	0.0076	0.0063	0.0051	0.0042	0.0034	0.0027	0.0022	0.0014	0.0009	0.0005
70	0.0067	0.0054	0.0043	0.0035	0.0027	0.0022	0.0017	0.0010	0.0006	0.0004
75	0.0059	0.0047	0.0037	0.0029	0.0022	0.0017	0.0013	0.0008	0.0004	0.0002
80	0.0052	0.0040	0.0031	0.0024	0.0018	0.0014	0.0010	0.0006	0.0003	0.0002
85	0.0046	0.0035	0.0026	0.0020	0.0015	0.0011	0.0008	0.0004	0.0002	0.0001
90	0.0040	0.0030	0.0023	0.0017	0.0012	0.0009	0.0006	0.0003	0.0002	0.0001
95	0.0036	0.0026	0.0019	0.0014	0.0010	0.0007	0.0005	0.0002	0.0001	0.0000
100	0.0032	0.0023	0.0016	0.0012	0.0008	0.0006	0.0004	0.0002	0.0001	0.0000

USE OF TABLE F

Table F gives the value of a_n of Eq. 34, Table III, for given values of i, h, and n.

When payment is made at the end of each year and interest is compounded annually.

Example 1.—What end-of-year annuity will be produced for 10 years by a principal of \$1, if interest is 4 per cent. compounded annually? Opposite "10" and under "4 per cent.," Table F, we find \$0.1233, which is the annuity sought.

Example 2.—What end-of-year payment for a period of five years is the equivalent of \$1000 paid now, if interest is 7 per cent. compounded annually? Opposite "5" and under "7 per cent.," Table F, we find \$0.2439, which multiplied by 1000 gives \$243.90±, which is the quantity sought.

When the payment is made at the end of each interest period which is less than one year.

Example 3.—A present debt of \$10,000 is to be paid in 20 equal semi-annual payments, one at the end of each half-year. If interest is 4 per cent., compounded semi-annually, what should each payment be? $i/h = 0.04/2 = 0.02$, $hn = 2 \times 10 = 20$. Opposite "20" and under "2 per cent.," Table F, we find 0.0612, which multiplied by 10,000 gives \$612±, which is the quantity sought.

Table F.—Annuity (a_n) Derived From $1 Principal

The end-of-year annuity produced for n years by a principal of $1 when interest is compounded annually at

n	2%	2-½%	3%	3-½%	4%	4-½%	5%	6%	7%	8%
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700	1.0800
2	0.5150	0.5188	0.5226	0.5264	0.5302	0.5340	0.5378	0.5454	0.5531	0.5608
3	0.3468	0.3501	0.3535	0.3569	0.3603	0.3638	0.3672	0.3741	0.3811	0.3880
4	0.2626	0.2658	0.2690	0.2723	0.2755	0.2787	0.2820	0.2886	0.2952	0.3019
5	0.2122	0.2152	0.2184	0.2215	0.2246	0.2278	0.2310	0.2374	0.2439	0.2505
6	0.1785	0.1815	0.1846	0.1877	0.1908	0.1939	0.1970	0.2034	0.2098	0.2163
7	0.1545	0.1575	0.1605	0.16[illegible]5	0.1666	0.1697	0.1728	0.1791	0.1856	0.1921
8	0.1365	0.1395	0.1425	0.1455	0.1485	0.1516	0.1547	0.1610	0.1675	0.1740
9	0.1225	0.1255	0.1284	0.1314	0.1345	0.1376	0.1407	0.1470	0.1535	0.1601
10	0.1113	0.1143	0.1172	0.1202	0.1233	0.1264	0.1295	0.1359	0.1424	0.1490
11	0.1022	0.1051	0.1081	0.1111	0.1141	0.1172	0.1204	0.1268	0.1334	0.1401
12	0.0946	0.0975	0.1005	0.1035	0.1066	0.1097	0.1128	0.1193	0.1259	0.1327
13	0.0881	0.0910	0.0940	0.0971	0.1001	0.1033	0.1065	0.1130	0.1197	0.1265
14	0.0826	0.0855	0.0885	0.0916	0.0947	0.0978	0.1010	0.1076	0.1143	0.1213
15	0.0778	0.0808	0.0838	0.0868	0.0899	0.0931	0.0963	0.1030	0.1098	0.1168
16	0.0737	0.0766	0.0796	0.0827	0.0858	0.0890	0.0923	0.0990	0.1059	0.1130
17	0.0700	0.0729	0.0760	0.0790	0.0822	0.0854	0.0887	0.0954	0.1024	0.1096
18	0.0667	0.0697	0.0727	0.0758	0.0790	0.0822	0.0855	0.0924	0.0994	0.1067
19	0.0638	0.0668	0.0698	0.0729	0.0761	0.0794	0.0827	0.0896	0.0968	0.1041
20	0.0612	0.0641	0.0672	0.0704	0.0736	0.0769	0.0802	0.0872	0.0944	0.1019
21	0.0588	0.0618	0.0649	0.0680	0.0713	0.0746	0.0780	0.0850	0.0923	0.0998
22	0.0566	0.0596	0.0627	0.0659	0.0692	0.0725	0.0760	0.0830	0.0904	0.0980
23	0.0547	0.0577	0.0608	0.0640	0.0673	0.0707	0.0741	0.0813	0.0887	0.0964
24	0.0529	0.0559	0.0590	0.0623	0.0656	0.0690	0.0725	0.0797	0.0872	0.0950
25	0.0512	0.0543	0.0574	0.0607	0.0640	0.0674	0.0710	0.0782	0.0858	0.0937
26	0.0497	0.0528	0.0559	0.0592	0.0626	0.0660	0.0696	0.0769	0.0846	0.0925
27	0.0483	0.0514	0.0546	0.0579	0.0612	0.0647	0.0683	0.0757	0.0834	0.0914
28	0.0470	0.0501	0.0533	0.0566	0.0600	0.0635	0.0671	0.0746	0.0824	0.0905
29	0.0458	0.0489	0.0521	0.0554	0.0589	0.0624	0.0660	0.0736	0.0814	0.0896
30	0.0446	0.0478	0.0510	0.0544	0.0578	0.0614	0.0651	0.0726	0.0806	0.0888
31	0.0436	0.0467	0.0500	0.0534	0.0569	0.0604	0.0641	0.0718	0.0798	0.0881
32	0.0426	0.0458	0.0490	0.0524	0.0559	0.0596	0.0633	0.0710	0.0791	0.0875
33	0.0417	0.0449	0.0482	0.0516	0.0551	0.0587	0.0625	0.0703	0.0784	0.0869
34	0.0408	0.0440	0.0473	0.0508	0.0543	0.0580	0.0618	0.0696	0.0778	0.0863
35	0.0400	0.0432	0.0465	0.0500	0.0536	0.0573	0.0611	0.0690	0.0772	0.0858
36	0.0392	0.0425	0.0458	0.0493	0.0529	0.0566	0.0604	0.0684	0.0767	0.0853
37	0.0385	0.0417	0.0451	0.0486	0.0522	0.0560	0.0598	0.0679	0.0762	0.0849
38	0.0378	0.0411	0.0445	0.0480	0.0516	0.0554	0.0593	0.0674	0.0758	0.0845
39	0.0372	0.0404	0.0438	0.0474	0.0511	0.0549	0.0588	0.0669	0.0754	0.0842
40	0.0366	0.0398	0.0433	0.0468	0.0505	0.0543	0.0583	0.0665	0.0750	0.0839
41	0.0360	0.0393	0.0427	0.0463	0.0500	0.0539	0.0578	0.0661	0.0747	0.0836
42	0.0354	0.0387	0.0422	0.0458	0.0495	0.0534	0.0574	0.0657	0.0743	0.0833
43	0.0349	0.0382	0.0417	0.0453	0.0491	0.0530	0.0570	0.0653	0.0740	0.0830
44	0.0344	0.0377	0.0412	0.0449	0.0487	0.0526	0.0566	0.0650	0.0738	0.0028
45	0.0339	0.0373	0.0408	0.0445	0.0483	0.0522	0.0563	0.0647	0.0735	0.0826
46	0.0335	0.0368	0.0404	0.0441	0.0479	0.0518	0.0559	0.0644	0.0733	0.0824
47	0.0330	0.0364	0.0400	0.0437	0.0475	0.0515	0.0556	0.0641	0.0730	0.0822
48	0.0326	0.0360	0.0396	0.0433	0.0472	0.0512	0.0553	0.0639	0.0728	0.0820
49	0.0322	0.0356	0.0392	0.0430	0.0469	0.0509	0.0550	0.0637	0.0726	0.0819
50	0.0318	0.0353	0.0389	0.0426	0.0466	0.0506	0.0548	0.0634	0.0725	0.0817
55	0.0301	0.0337	0.0373	0.0412	0.0452	0.0494	0.0537	0.0625	0.0717	0.0812
60	0.0288	0.0324	0.0361	0.0401	0.0442	0.0485	0.0528	0.0619	0.0712	0.0808
65	0.0276	0.0313	0.0351	0.0392	0.0434	0.0477	0.0522	0.0614	0.0709	0.0805
70	0.0267	0.0304	0.0343	0.0385	0.0427	0.0472	0.0517	0.0610	0.0706	0.0804
75	0.0259	0.0297	0.0337	0.0379	0.0422	0.0467	0.0513	0.0608	0.0704	0.0802
80	0.0252	0.0290	0.0331	0.0274	0.0418	0.0464	0.0510	0.0606	0.0703	0.0802
85	0.0246	0.0285	0.0326	0.0370	0.0415	0.0461	0.0508	0.0604	0.0702	0.0801
90	0.0240	0.0280	0.0223	0.0367	0.0412	0.0459	0.0506	0.0603	0.0702	0.0801
95	0.0236	0.0276	0.0319	0.0364	0.0410	0.0457	0.0505	0.0602	0.0701	0.0800
100	0.0232	0.0273	0.0316	0.0362	0.0408	0.0456	0.0504	0.0602	0.0701	0.0800

Table G.—Number of Days in the Year up to and Including any Given Date

Day of month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept	Oct	Nov	Dec.	Day of month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29	....	88	119	149	180	210	241	272	302	333	363	29
30	30	.....	89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

For leap year add 1 to each number after the number for Feb. 28.

USE OF TABLE H

Example.—A note for $280.50, bearing 6 per cent. simple interest, is dated September 22, 1905 and falls due June 17, 1907. What is the simple interest on the note for the whole time?

Solution on the basis of a 365-day year:

September 22, 1905 to September 22, 1906.	1 year,
September 22, 1906 to June 17, 1907:	
Days in one year	365
September 22 is, of the year, day number (Table G)	265
Remaining days of 1906 (not a leap year)	100
Days, January 1 to June 17 inclusive (Table G) . .	168
Days, September 22, 1906 to June 17, 1907	268 days.
Whole time	1 year, 268 days.

Interest at 1 per cent. on $280.50 for 1 year.	$2 8050
Interest at 1 per cent. on 200.00 for 260 days (Table H)	1.4246
Interest at 1 per cent. on 80.00 for 260 days (Table H)	0.5698
Interest at 1 per cent. on 0.50 for 260 days (Table H)	0.0035
Interest at 1 per cent. on 200.00 for 8 days (Table H)	0 0438
Interest at 1 per cent. on 80.00 for 8 days (Table H)	0.0175
Interest at 1 per cent. on 0.50 for 8 days (Table H)	0.0001
Interest at 1 per cent. on $280.50 for 1 year, 268 days	$4.8643
which multiplied by.	6
gives the interest at 6 per cent. on $280.50 for 1 year 268 days.	$29.1859,

or, $29.19. *Answer.* (This is the same answer as was computed directly by formula in §8, using the 365-day year.)

Table H.—Simple Interest (365-day Year)
At 1 per cent. on

Days	$1000	$2000	$3000
1	0.02740	0.05479	0.08219
2	0.05479	0.10959	0.16438
3	0 08219	0 16438	0.24658
4	0 10959	0 21918	0.32877
5	0 13699	0.27397	0.41096
6	0.16438	0.32877	0.49315
7	0 19178	0.38356	0 57534
8	0 21918	0.43836	0.65753
9	0.24658	0.49315	0.73973
10	0 27397	0.54795	0.82192
20	0 54795	1.09589	1.64384
30	0 82192	1 64384	2.46575
40	1 09589	2 19178	3.28767
50	1 36986	2.73973	4.10959
60	1 64384	3.28767	4.93151
70	1 91781	3 83562	5.75342
80	2.19178	4.38356	6.57534
90	2.46575	4.93151	7.39726
100	2 73973	5.47945	8.21918
110	3 01370	6.02740	9.04110
120	3 28767	6.57534	9.86301
130	3.56164	7.12329	10.68493
140	3 83562	7.67123	11 50685
150	4.10959	8.21918	12 32877
160	4 38356	8.76712	13 15068
170	4 65753	9.31507	13.97260
180	4.93151	9.86301	14 79452
190	5 20548	10.41096	15.61644
200	5.47945	10.95890	16 43836
210	5.75342	11.50685	17.26027
220	6.02740	12.05479	18.08219
230	6.30137	12.60274	18 90411
240	6.57534	13.15068	19 72603
250	6.84932	13.69863	20.54795
260	7.12329	14.24658	21.36986
270	7.39726	14.79452	22.19178
280	7.67123	15.34247	23.01370
290	7.94521	15.89041	23 83562
300	8.21918	16.43836	24 65753
310	8.49315	16.98630	25 47945
320	8.76712	17 53425	26.30137
330	9.04110	18.08219	27.12329
340	9.31507	18.63014	27 94521
350	9.58904	19 17808	28 76712
360	9.86301	19.72603	29 58904

Table H.—Simple Interest (365-day Year) (*Continued*)
At 1 per cent. on

Days	$4000	$5000	$6000
1	0 10959	0 13699	0.16438
2	0 21918	0.27397	0 32877
3	0.32877	0 41096	0.49315
4	0.43836	0.54795	0.65753
5	0 54795	0.68493	0.82192
6	0.65753	0.82192	0 98630
7	0 76712	0.95890	1.15068
8	0 87671	1.09589	1 31507
9	0.98630	1.23288	1 47945
10	1.09589	1.36986	1.64384
20	2 19178	2.73973	3 28767
30	3.28767	4 10959	4.93151
40	4.38356	5 47945	6.57534
50	5.47945	6 84932	8 21918
60	6.57534	8.21918	9 86301
70	7.67123	9.58904	11 50685
80	8.76712	10 95890	13.15068
90	9.86301	12.32877	14 79452
100	10 95890	13 69863	16 43836
110	12 05479	15 06849	18 08219
120	13.15068	16.43836	19 72603
130	14.24658	17.80822	21 36986
140	15.34247	19 17808	23 01370
150	16.43836	20 54795	24 65753
160	17.53425	21.91781	26.30137
170	18.63014	23 28767	27 94521
180	19 72603	24 65753	29.58904
190	20.82192	26.02740	31.23288
200	21.91781	27 39726	32 87671
210	23.01370	28 76712	34 52055
220	24.10959	30.13699	36 16438
230	25 20548	31.50685	37.80822
240	26.30137	32 87671	39 45205
250	27.39726	34.24658	41 09589
260	28 49315	35 61644	42.73973
270	29 58904	36 98630	44.38356
280	30.68493	38.35616	46 02740
290	31.78082	39.72603	47.67123
300	32.87671	41.09589	49 31507
310	33.97260	42 46575	50 95890
320	35 06849	43.83562	52.60274
330	36.16438	45 20548	54.24657
340	37 26027	46.57534	55.89041
350	38.35616	47.94521	57.53425
360	39.45205	49.31507	59.17808

Table H.—Simple Interest (365-day Year)

At 1 per cent. on

Days	$1000	$2000	$3000
1	0 02740	0 05479	0.08219
2	0 05479	0.10959	0.16438
3	0.08219	0 16438	0.24658
4	0 10959	0.21918	0.32877
5	0.13699	0 27397	0.41096
6	0.16438	0.32877	0.49315
7	0 19178	0.38356	0.57534
8	0 21918	0 43836	0.65753
9	0.24658	0.49315	0.73973
10	0.27397	0.54795	0.82192
20	0 54795	1 09589	1.64384
30	0 82192	1.64384	2.46575
40	1 09589	2.19178	3.28767
50	1 36986	2.73973	4.10959
60	1 64384	3.28767	4.93151
70	1 91781	3.83562	5.75342
80	2.19178	4.38356	6.57534
90	2 46575	4.93151	7.39726
100	2 73973	5.47945	8.21918
110	3.01370	6.02740	9.04110
120	3 28767	6.57534	9.86301
130	3 56164	7.12329	10 68493
140	3.83562	7.67123	11 50685
150	4.10959	8.21918	12.32877
160	4.38356	8.76712	13 15068
170	4.65753	9.31507	13.97260
180	4 93151	9.86301	14 79452
190	5.20548	10.41096	15 61644
200	5 47945	10.95890	16 43836
210	5 75342	11.50685	17.26027
220	6.02740	12.05479	18.08219
230	6 30137	12.60274	18 90411
240	6.57534	13.15068	19.72603
250	6.84932	13.69863	20 54795
260	7.12329	14.24658	21 36986
270	7.39726	14.79452	22.19178
280	7.67123	15.34247	23 01370
290	7.94521	15.89041	23 83562
300	8.21918	16.43836	24.65753
310	8.49315	16.98630	25 47945
320	8.76712	17.53425	26 30137
330	9.04110	18.08219	27.12329
340	9.31507	18.63014	27.94521
350	9 58904	19.17808	28 76712
360	9.86301	19.72603	29 58904

Table H.—Simple Interest (365-day Year) (*Continued*)

At 1 per cent. on

Days	$4000	$5000	$6000
1	0 10959	0 13699	0 16438
2	0 21918	0.27397	0.32877
3	0.32877	0 41096	0 49315
4	0.43836	0.54795	0 65753
5	0 54795	0 68493	0.82192
6	0 65753	0.82192	0 98630
7	0 76712	0.95890	1.15068
8	0 87671	1.09589	1 31507
9	0.98630	1.23288	1 47945
10	1 09589	1.36986	1.64384
20	2 19178	2.73973	3 28767
30	3 28767	4 10959	4.93151
40	4 38356	5.47945	6 57534
50	5.47945	6 84932	8 21918
60	6 57534	8.21918	9 86301
70	7 67123	9 58904	11 50685
80	8.76712	10 95890	13 15068
90	9 86301	12.32877	14.79452
100	10 95890	13.69863	16 43836
110	12.05479	15.06849	18 08219
120	13.15068	16 43836	19 72603
130	14.24658	17.80822	21 36986
140	15.34247	19 17808	23 01370
150	16.43836	20.54795	24.65753
160	17.53425	21.91781	26.30137
170	18.63014	23.28767	27.94521
180	19.72603	24 65753	29 58904
190	20 82192	26 02740	31.23288
200	21.91781	27 39726	32 87671
210	23.01370	28.76712	34 52055
220	24 10959	30.13699	36 16438
230	25.20548	31.50685	37.80822
240	26 30137	32 87671	39.45205
250	27.39726	34.24658	41.09589
260	28.49315	35.61644	42 73973
270	29.58904	36.98630	44.38356
280	30 68493	38 35616	46 02740
290	31.78082	39.72603	47 67123
300	32.87671	41 09589	49 31507
310	33.97260	42 46575	50 95890
320	35.06849	43.83562	52 60274
330	36 16438	45.20548	54.24657
340	37.26027	46.57534	55.89041
350	38.35616	47.94521	57.53425
360	39.45205	49.31507	59.17808

Table H.—Simple Interest (365-day Year) (*Continued*)

At 1 per cent. on

Days	$7000	$8000	$9000
1	0.19178	0 21918	0 24658
2	0 38356	0.43836	0.49315
3	0 57534	0 65753	0 73973
4	0.76712	0.87671	0 98630
5	0.95890	1 09589	1.23288
6	1 15068	1 31507	1 47945
7	1.34247	1 53425	1.72603
8	1.53425	1.75342	1 97260
9	1.72603	1.97260	2.21918
10	1.91781	2.19178	2 46575
20	3.83562	4.38356	4.93151
30	5.75342	6 57534	7.39726
40	7.67123	8 76712	9.86301
50	9.58904	10.95890	12.32877
60	11.50685	13.15068	14 79452
70	13.42466	15.34247	17 26027
80	15.34247	17.53425	19.72603
90	17.26027	19.72603	22.19178
100	19.17808	21.91781	24 65753
110	21.09589	24.10959	27.12329
120	23.01370	26 30137	29.58904
130	24.93151	28 49315	32 05479
140	26.84931	30 68493	34 52055
150	28.76712	32 87671	36.98630
160	30.68493	35.06849	39 45205
170	32.60274	37 26027	41.91781
180	34.52055	39.45205	44.38356
190	36 43836	41 64384	46.84931
200	38.35616	43.83562	49.31507
210	40.27397	46.02740	51.78082
220	42 19178	48 21918	53.24657
230	44.10959	50 41096	56.71233
240	46.02740	52.60274	59 17808
250	47 94521	54.79452	61 64384
260	49 86301	56.98630	64.10959
270	51.78082	59.17808	66 57534
280	53.69863	61.36986	69 04110
290	55.61644	63 56164	71.50685
300	57.53425	65.75342	73.97260
310	59.45205	67.94520	76 43836
320	61.36986	70.13699	78 90411
330	63.28767	72.32877	81.36986
340	65.20548	74.52055	83.83562
350	67.12329	76 71233	86.30137
360	69.04110	78.90411	88.76712

USE OF TABLE J

Example.—A note for $280.50 bearing 6 per cent. simple interest is dated Sept. 22, 1905 and falls due June 17, 1907. What is the simple interest on the note for the whole time?

Solution on the basis of a 360-day year:

	Year	Month	Day
June 17, 1907...........	1907	6	17
Sept. 22, 1905..........	1905	9	22
Time..................	1 year	8 months	25 days

Interest at 1 per cent. on $280.50 for 1 year............	$2.8050
Interest at 1 per cent. on 200.00 for 8 months (Table J)	1.3333
Interest at 1 per cent. on 80.00 for 8 months (Table J)	0.5333
Interest at 1 per cent. on 0.50 for 8 months (Table J)	0.0033
Interest at 1 per cent. on 200.00 for 25 days (Table J)	0.1388
Interest at 1 per cent. on 80.00 for 25 days (Table J)	0.0556
Interest at 1 per cent. on 0.50 for 25 days (Table J)	0.0003
Interest at 1 per cent. on $280.50 for 1 year, 8 months, 25 days..................................	$4 8696
which multiplied by...........................	6
gives the interest at 6 per cent. on $280.50 for 1 year, 8 months, 25 days..................	$29.2176,

or, $29.22. *Answer.* (This answer is the same as that computed by formula for 360-day year in §8.)

Table J.—Simple Interest (360-day Year)

At 1 per cent. on

Days	$1000	$2000	$3000
1	0 02777	0 05555	0.08333
2	0 05555	0 11111	0.16666
3	0.08333	0.16666	0.25000
4	0 11111	0.22222	0.33333
5	0.13888	0.27777	0.41666
6	0 16666	0.33333	0 50000
7	0.19444	0.38888	0 58333
8	0 22222	0.44444	0.66666
9	0 25000	0.50000	0 75000
10	0 27777	0.55555	0.83333
11	0.30555	0 61111	0 91666
12	0.33333	0 66666	1.00000
13	0.36111	0.72222	1.08333
14	0.38888	0.77777	1 16666
15	0.41666	0.83333	1 25000
16	0.44444	0 88888	1.33333
17	0.47222	0 94444	1.41666
18	0.50000	1.00000	1.50000
19	0.52777	1 05555	1.58333
20	0.55555	1.11111	1.66666
21	0 58333	1.16666	1.75000
22	0 61111	1.22222	1.83333
23	0 63888	1.27777	1 91666
24	0.66666	1.33333	2.00000
25	0.69444	1.38888	2.08333
26	0.72222	1.44444	2.16666
27	0.75000	1.50000	2.25000
28	0.77777	1.55555	2.33333
29	0 80555	1.61111	2.41666
Months			
1	0.83333	1.66666	2.50000
2	1 66666	3.33333	5.00000
3	2.50000	5.00000	7.50000
4	3 33333	6.66666	10.00000
5	4.16666	8.33333	12.50000
6	5 00000	10.00000	15.00000
7	5.83333	11 66666	17.50000
8	6.66666	13 33333	20.00000
9	7 50000	15 00000	22.50000
10	8.33333	16 66666	25 00000
11	9.16666	18.33333	27.50000
Year			
1	10.00000	20.00000	30.00000

Table J.—Simple Interest (360-day Year) (*Continued*)

At 1 per cent. on

Days	$4000	$5000	$6000
1	0.11111	0.13888	0 16666
2	0.22222	0.27777	0 33333
3	0 33333	0.41666	0.50000
4	0 44444	0 55555	0 66666
5	0.55555	0 69444	0.83333
6	0.66666	0 83333	1.00000
7	0 77777	0.97222	1.16666
8	0 88888	1.11111	1.33333
9	1 00000	1.25000	1 50000
10	1 11111	1.38888	1.66666
11	1 22222	1 52777	1.83333
12	1 33333	1.66666	2.00000
13	1.44444	1.80555	2 16666
14	1 55555	1.94444	2.33333
15	1.66666	2 08333	2 50000
16	1.77777	2.22222	2 66666
17	1 88888	2 36111	2 83333
18	2.00000	2 50000	3 00000
19	2 11111	2 63888	3 16666
20	2 22222	2 77777	3 33333
21	2.33333	2 91666	3 50000
22	2 44444	3.05555	3 66666
23	2.55555	3 19444	3.83333
24	2 66666	3.33333	4 00000
25	2.77777	3.47222	4 16666
26	2 88888	3 61111	4 33333
27	3 00000	3 75000	4 50000
28	3.11111	3.88888	4 66666
29	3.22222	4 02777	4 83333
Months			
1	3.33333	4.16666	5 00000
2	6.66666	8.33333	10.00000
3	10.00000	12 50000	15.00000
4	13 33333	16 66666	20 00000
5	16 66666	20.83333	25.00000
6	20.00000	25.00000	30 00000
7	23.33333	29.16666	35 00000
8	26 66666	33 33333	40.00000
9	30 00000	37 50000	45 00000
10	33 33333	41 66666	50.00000
11	36 66666	45.83333	55 00000
Year			
1	40 00000	50.00000	60 00000

Table J.—Simple Interest (360-day Year) (*Continued*)

At 1 per cent. on

Days	$7000	$8000	$9000
1	0.19444	0 22222	0.25000
2	0.38888	0.44444	0 50000
3	0.58333	0 66666	0.75000
4	0.77777	0 88888	1.00000
5	0.97222	1.11111	1.25000
6	1.16666	1.33333	1.50000
7	1.36111	1 55555	1.75000
8	1.55555	1.77777	2.00000
9	1.75000	2.00000	2 25000
10	1.94444	2.22222	2.50000
11	2.13888	2.44444	2.75000
12	2.33333	2 66666	3.00000
13	2.52777	2.88888	3 25000
14	2 72222	3.11111	3.50000
15	2.91666	3.33333	3.75000
16	3.11111	3.55555	4.00000
17	3.30555	3.77777	4.25000
18	3.50000	4.00000	4.50000
19	3.69333	4.22222	4.75000
20	3 88888	4.44444	5.00000
21	4.08333	4.66666	5.25000
22	4 27777	4.88888	5.50000
23	4 47222	5.11111	5.75000
24	4.66666	5.33333	6.00000
25	4.86111	5 55555	6.25000
26	5.05555	5.77777	6.50000
27	5.25000	6.00000	6 75000
28	5.44444	6.22222	7.00000
29	5.63888	6.44444	7.25000
Months			
1	5 83333	6 66666	7.50000
2	11 66666	13.33333	15.00000
3	17.50000	20 00000	22.50000
4	23.33333	26 66666	30.00000
5	29 16666	33.33333	37.50000
6	35.00000	40 00000	45 00000
7	40.83333	46.66666	52.50000
8	46.66666	53.33333	60.00000
9	52.50000	60 00000	67 50000
10	58.33333	66 66666	75.00000
11	64.16666	73.33333	82.50000
Year			
1	70 00000	80.00000	90.00000

INDEX

In this index, numbers refer to pages. The abbreviation "(ref.)," prefixed to a number, shows that on the page indicated there is a reference to some other publication. The index gives the page on which the meaning of each letter of the notation of the book is stated, except in a few cases in which the index gives the meaning directly.